Online algorithms
Search advertising
Online algorithms

- Classic model of algorithms
  - You get to see the entire input, then compute some function of it
  - In this context, “offline algorithm”

- Online algorithm
  - You get to see the input one piece at a time, and need to make irrevocable decisions along the way

- Similar to data stream models
Example: Bipartite matching

![Bipartite matching diagram]

- Girls: 1, 2, 3, 4
- Boys: a, b, c, d

Each girl is matched with a boy, as shown in the diagram.
Example: Bipartite matching

M = {(1,a),(2,b),(3,d)} is a matching
Cardinality of matching = |M| = 3
Example: Bipartite matching

M = \{(1,c),(2,b),(3,d),(4,a)\} is a perfect matching
Matching Algorithm

- Problem: Find a maximum-cardinality matching for a given bipartite graph
  - A perfect one if it exists
- There is a polynomial-time offline algorithm (Hopcroft and Karp 1973)
- But what if we don’t have the entire graph upfront?
Online problem

- Initially, we are given the set Boys
- In each round, one girl’s choices are revealed
- At that time, we have to decide to either:
  - Pair the girl with a boy
  - Don’t pair the girl with any boy
- Example of application: assigning tasks to servers
Online problem

(1, a)  
(2, b)  
(3, d)
Greedy algorithm

- Pair the new girl with any eligible boy
  - If there is none, don’t pair girl
- How good is the algorithm?
Competitive Ratio

For input $I$, suppose greedy produces matching $M_{\text{greedy}}$ while an optimal matching is $M_{\text{opt}}$

Competitive ratio = $\min_{\text{all possible inputs } I} \left( \frac{|M_{\text{greedy}}|}{|M_{\text{opt}}|} \right)$
Analyzing the greedy algorithm

- Consider the set $G$ of girls matched in $M_{opt}$ but not in $M_{greedy}$
- Then it must be the case that every boy adjacent to girls in $G$ is already matched in $M_{greedy}$
- There must be at least $|G|$ such boys
  - Otherwise the optimal algorithm could not have matched all the $G$ girls
- Therefore
  $$|M_{greedy}| \cdot |G| = |M_{opt} - M_{greedy}|$$
  $$|M_{greedy}| / |M_{opt}| \leq 1/2$$
Worst-case scenario

(1,a)
(2,b)
History of web advertising

- Banner ads (1995-2001)
  - Initial form of web advertising
  - Popular websites charged X$ for every 1000 “impressions” of ad
    - Called “CPM” rate
    - Modeled similar to TV, magazine ads
  - Untargeted to demographically targeted
  - Low clickthrough rates
    - low ROI for advertisers
Performance-based advertising

- Introduced by Overture around 2000
  - Advertisers “bid” on search keywords
  - When someone searches for that keyword, the highest bidder’s ad is shown
  - Advertiser is charged only if the ad is clicked on

- Similar model later adopted by Google with some changes
  - Called “Adwords”
Ads vs. search results

GEICO Car Insurance. Get an auto insurance quote and save today...
GEICO auto insurance, online car insurance quote, motorcycle insurance quote, online insurance sales and service from a leading insurance company.
www.geico.com/ - 21k - Sep 22, 2005 - Cached - Similar pages
Auto Insurance - Buy Auto Insurance
Contact Us - Make a Payment
More results from www.geico.com »

Geico, Google Settle Trademark Dispute
The case was resolved out of court, so advertisers are still left without legal guidance on use of trademarks within ads or as keywords.

Google and GEICO settle AdWords dispute | The Register
Google and car insurance firm GEICO have settled a trademark dispute over ... Car insurance firm GEICO sued both Google and Yahoo subsidiary Overture in ...
www.theregister.co.uk/2005/09/09/google_geico_settlement/ - 21k - Cached - Similar pages

GEICO v. Google
... involving a lawsuit filed by Government Employees Insurance Company (GEICO). GEICO
has filed suit against two major Internet search engine operators, ...
www.consumeraffairs.com/news04/geico_google.html - 19k - Cached - Similar pages

Sponsored Links
Great Car Insurance Rates
Simplify Buying insurance at Safeco
See Your Rate with an Instant Quote
www.Safeco.com

Free Insurance Quotes
Fill out one simple form to get multiple quotes from local agents.
www.HometownQuotes.com

5 Free Quotes, 1 Form.
Get 5 Free Quotes In Minutes!
You Have Nothing To Lose. It's Free
sayyesoftware.com/Insurance
Missouri
Web 2.0

- Performance-based advertising works!
  - Multi-billion-dollar industry
- Interesting problems
  - What ads to show for a search?
  - If I’m an advertiser, which search terms should I bid on and how much to bid?
Adwords problem

- A stream of queries arrives at the search engine
  - q1, q2,...
- Several advertisers bid on each query
- When query $q_i$ arrives, search engine must pick a subset of advertisers whose ads are shown
- Goal: maximize search engine’s revenues
- Clearly we need an online algorithm!
Greedy algorithm

- Simplest algorithm is greedy
- It’s easy to see that the greedy algorithm is actually optimal!
Complications (1)

- Each ad has a different likelihood of being clicked
  - Advertiser 1 bids $2, click probability = 0.1
  - Advertiser 2 bids $1, click probability = 0.5
  - Clickthrough rate measured historically

- Simple solution
  - Instead of raw bids, use the “expected revenue per click”
Complications (2)

- Each advertiser has a limited budget
  - Search engine guarantees that the advertiser will not be charged more than their daily budget
Simplified model (for now)

- Assume all bids are 0 or 1
- Each advertiser has the same budget B
- One advertiser per query
- Let’s try the greedy algorithm
  - Arbitrarily pick an eligible advertiser for each keyword
Bad scenario for greedy

- Two advertisers A and B
- A bids on query x, B bids on x and y
- Both have budgets of $4
- Query stream: x x x y y y y
  - Worst case greedy choice: B B B B____
  - Optimal: A A A A B B B B
  - Competitive ratio = ½
- Simple analysis shows this is the worst case
BALANCE algorithm [MSVV]

- [Mehta, Saberi, Vazirani, and Vazirani]
- For each query, pick the advertiser with the largest unspent budget
  - Break ties arbitrarily
Example: BALANCE

- Two advertisers A and B
- A bids on query x, B bids on x and y
- Both have budgets of $4
- Query stream: xxxxyyyyy
- BALANCE choice: ABABBB__
  - Optimal: AAAABBBBB
- Competitive ratio = \( \frac{3}{4} \)
Analyzing BALANCE (1)

- Consider simple case: two advertisers, P and Q, each with budget B (assume $B \leq 1$).
- Assume optimal solution exhausts both advertisers’ budgets.
  - OPT = 2B
- BALANCE must exhaust at least one advertiser’s budget.
  - If not, we can allocate more queries.
  - Assume BALANCE exhausts Q’s budget, but allocates x queries fewer than the optimal.
  - BAL = 2B - x
Analyzing Balance

Queries allocated to $A_1$ in optimal solution
Queries allocated to $A_2$ in optimal solution

Opt revenue = 2B
Balance revenue = 2B-x = B+y

We have $y \geq x$
Balance revenue is minimum for $x=y=B/2$
Minimum Balance revenue = 3B/2
Competitive Ratio = 3/4
Analyzing BALANCE (2)

- Three types of queries:
  (A) P is the only bidder
  (B) Q is the only bidder
  (C) P and Q both bid

- Since Q’s budget is exhausted but P’s is not, and we couldn’t allocate x queries, they must be of type C
Analyzing BALANCE (3)

- BALANCE allocates at least $x$ Type C queries to Q
  - In the Optimal, these were assigned to P
- Consider the last Type C query assigned to Q
  - At this point, Q’s leftover budget was greater than P’s
  - So P’s allocation was at least $x$
- So we have $\text{BAL} \geq B + x$
Analyzing BALANCE (4)

We now have:
BAL = 2B – x
BAL ≥ B + x

The minimum value of BAL is obtained when x = B/2
BAL = 3B/2
OPT = 2B
So BAL/OPT = 3/4
General Result

- In the general case, worst competitive ratio of BALANCE is $1 - \frac{1}{e} = \text{approx. } 0.63$
- Interestingly, no online algorithm has a better competitive ratio
- Won’t go through the details here, but let’s see the worst case that gives this ratio
Worst case for BALANCE

- N advertisers, each with budget B \( \geq N \geq 1 \)
- NB queries appear in N rounds of B queries each
- Round 1 queries: bidders \( A_1, A_2, \ldots, A_N \)
- Round 2 queries: bidders \( A_2, A_3, \ldots, A_N \)
- Round i queries: bidders \( A_i, \ldots, A_N \)
- Optimum allocation: allocate round i queries to \( A_i \)
  - Optimum revenue NB
After $k$ rounds, sum of allocations to each of bins $A_k, \ldots, A_N$ is 
$S_k = S_{k+1} = \ldots = S_N = \sum_{1 \leq i \leq k} B/(N-i+1)$

If we find the smallest $k$ such that $S_k \leq B$, then after $k$ rounds 
we cannot allocate any queries to any advertiser
BALANCE analysis

\[ \begin{align*}
B/1 & \quad B/2 & \quad B/3 & \ldots & \quad B/(N-k+1) & \ldots & \quad B/(N-1) & \quad B/N \\
S_1 & \quad S_2 & \quad S_k = B
\end{align*} \]

\[ \begin{align*}
1/1 & \quad 1/2 & \quad 1/3 & \ldots & \quad 1/(N-k+1) & \ldots & \quad 1/(N-1) & \quad 1/N \\
S_1 & \quad S_2 & \quad S_k = 1
\end{align*} \]
BALANCE analysis

Fact: $H_n = \sum_{1 \leq i \leq n} 1/i = $ approx. $\log(n)$ for large $n$

- Result due to Euler

\[ 1/1 \quad 1/2 \quad 1/3 \quad \ldots \quad 1/(N-k+1) \quad \ldots \quad 1/(N-1) \quad 1/N \]

$S_k = 1$ implies $H_{N-k} = \log(N) - 1 = \log(N/e)$

$N-k = N/e$

$k = N(1-1/e)$
平衡分析

- 所以，在前 $N(1-1/e)$ 轮中，我们无法将查询分配给任何广告商
- 收入 $= BN(1-1/e)$
- 竞争比 $= 1-1/e$
General version of problem

- Arbitrary bids, budgets
- Consider query q, advertiser i
  - Bid = $x_i$
  - Budget = $b_i$
- BALANCE can be terrible
  - Consider two advertisers $A_1$ and $A_2$
  - $A_1$: $x_1 = 1$, $b_1 = 110$
  - $A_2$: $x_2 = 10$, $b_2 = 100$
Generalized BALANCE

- Arbitrary bids; consider query q, bidder i
  - Bid = $x_i$
  - Budget = $b_i$
  - Amount spent so far = $m_i$
  - Fraction of budget left over $f_i = 1 - m_i/b_i$
  - Define $\psi_i(q) = x_i(1-e^{-f_i})$
- Allocate query q to bidder i with largest value of $\psi_i(q)$
- Same competitive ratio $(1-1/e)$