Numerical Analysis Project.
Manuscript NA-87-06

June 1987

Numerical Assessment of the Validity of Two-dimensional Plate Models

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Acknowledgment

This work has been realized at the Computer Science Department of Stanford University during my sabbatical leave from ESIEE. I wish to thank both institutions and specially Gene Golub for his warm welcome.

I wish also to thank Philippe Ciarlet and Michael Vogelius for many fruitful conversations.

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0. Introduction

The objective of this paper is to verify numerically the convergence of the solution to the three-dimensional problem of a clamped plate towards the solution to the corresponding "limit" two-dimensional problem when the thickness of the plate goes to zero.

Standard finite elements discretization of the three-dimensional problem fails to show this convergence [6] as they lead to ill-conditioned linear systems when the discretization parameter is of the order of the thickness. We will therefore use a spectral approximation of the solution of the three-dimensionalproblem.

First, we shall review the three-dimensional and two-dimensional linear models of a clampled plate and give the convergence results obtained by P.-G. Ciarlet and P. Destuynder [1], [2].

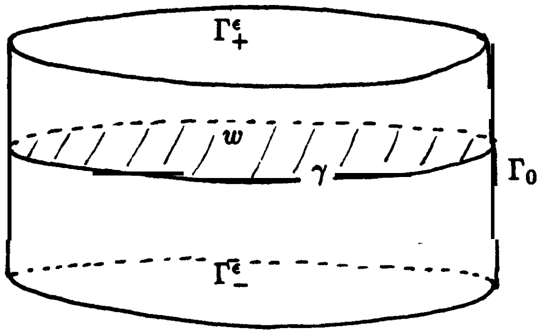
Then we will discuss two kinds of spectral approximations: the Galerkin and Tau approximations.

Finally we give the numerical results obtained by Tau approximation.

1. The Physical Problem

1.1. The Three-dimensional Clamped Plate Model

Let w be a bounded open set in R^2 with Lipschitz boundary 7. The plate occupies the volume $\overline{\Omega}^\epsilon$ where Ω^ϵ is defined by: $\Omega^\epsilon = w \times]-\epsilon, +\epsilon[$. Its boundary I” is the union of the lateral boundary Γ_0^ϵ = 7 x $[-\epsilon, +\epsilon]$, the upper boundary Γ_+^ϵ = w x $\{+\epsilon\}$, and the lower boundary I”, = w x $\{-\epsilon\}$, $\Gamma^\epsilon = \Gamma_0^\epsilon \cup \Gamma_+^\epsilon \cup \Gamma_-^\epsilon$.



Let $x^\epsilon = (x_i^\epsilon)_{1\leq i\leq 3}$ denote a generic point of the body $\overline{\Omega}^\epsilon$ and $\partial_i u^\epsilon = \frac{\partial u^\epsilon}{\partial x_i^\epsilon}$. Let $u^\epsilon = (u_i^\epsilon)_{1\leq i\leq 3}$: $\overline{\Omega}^\epsilon \rightarrow R^3$ denote the displacement field and $\sigma^\epsilon = (\sigma_{ij}^\epsilon)_{1\leq i,j\leq 3}$: $\overline{\Omega}^\epsilon \rightarrow S^3$ the second Piola-Kirchhoff stress field. The plate is subjected to body forces $f^\epsilon = (f_i^\epsilon)_{1\leq i\leq 3}$: $\Omega^\epsilon \rightarrow R^3$ and to surface forces $g^{\epsilon\pm} = (g_i^{\epsilon\pm})_{1\leq i\leq 3}$: $\Gamma_+^\epsilon \cup F_-^\epsilon \rightarrow R?$

We shall hereafter suppose that the applied forces are small enough to allow the use of a linear model of elastic material. In this case the equilibrium equations are

$$\begin{cases} -\mathrm{div}^\epsilon \sigma^\epsilon = f^\epsilon & \text{in } \Omega^\epsilon, \\ \sigma_{i3}^\epsilon = g_i^{\epsilon\pm} & \text{on } \Gamma_\pm^\epsilon. \end{cases}$$

The condition under which the plate is clamped is expressed by:

$$u^\epsilon = 0 \qquad \text{on } \Gamma_0^\epsilon.$$

For an isotropic, homogeneous, linearly elastic material the *constitutive* equations on Ω^ϵ are:

$$\sigma_{ij}^\epsilon = \lambda \sum_{k=1}^{3} e_{kk}^\epsilon \delta_{ij} + 2\mu e_{ij}^\epsilon, \qquad 1 \leq i, j \leq 3,$$

λ and μ are the Lamé’s constants of the material and the Green-Saint Venant tensor e_{ij}^ϵ is related to the displacement by:

$$2e_{ij}^\epsilon = \partial_i^\epsilon u_j^\epsilon + \partial_j^\epsilon u_i^\epsilon, \qquad 1 \leq i, j \leq 3.$$

The Lamé’s constants λ and μ are related to the Young modulus E and Poisson ratio ν by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad 2\mu = \frac{E}{1+\nu}.$$

1.2. The Two-dimensional Clamped Plate Model

With the same notation as in the first section, the Kirchhoff-Love model of a clamped plate is given by the following partial differential equation.

$$\begin{cases} \frac{2E\epsilon^3}{3(1-\nu^2)}\Delta^2\zeta = \int_{-\epsilon}^{\epsilon} f_3^\epsilon(x^\epsilon)\,dx_3^\epsilon + g_3^{\epsilon+}(x_1^\epsilon, x_2^\epsilon) + g_3^{\epsilon-}(x_1^\epsilon, x_2^\epsilon) & \text{in } \omega, \\ \zeta = \partial_\nu\zeta = 0 & \text{on 7.} \end{cases}$$

where ζ is the vertical component of the displacement.

In order to compare the behavior of the solutions to the two-dimensional model and to the three-dimensional model when the thickness (2ϵ) goes to zero, it is convenient to introduce a fixed set Ω independent of ϵ, Ω = ox$]-1,+1[$.

1.3. The Fixed Set Ω

Let $\Omega = \omega\times]-1,+1[$ be a fixed domain whose boundary Γ is the union of the lateral boundary $\Gamma^0 =$ 7 $\times[-1,+1]$ and of the upper and lower boundaries $\Gamma_\pm$ = w x {fl}. The relationship between a point $x^\epsilon = (x_1^\epsilon, x_2^\epsilon, x_3^\epsilon) \in \Omega^\epsilon$ and the corresponding-point 2 = $(x_1, x_2, x_3) \in \Omega$ is

$$x_1^\epsilon = x_1, \quad x_2^\epsilon = x_2, \quad x_3^\epsilon = \epsilon x_3 .$$

Let $u(\epsilon)$ be the displacement field of the body occupying the volume $\overline{\Omega}$. Let us make the following assumptions on the the displacement u for all $x^\epsilon \leftrightarrow$ 2:

$$\begin{cases} u_\alpha^\epsilon(x^\epsilon) = \epsilon^2 u_\alpha(\epsilon, x), & 1 \le \alpha \le 2 \\ u_3^\epsilon(x^\epsilon) = \epsilon\, u_3(\epsilon, \text{2)} \end{cases}$$

and on *the* applied *forces:* the horizontal components of the body and the surface forces are equal to zero and we suppose that there exist f_3 and g_3 independent of 23 such that for all $x^\epsilon \leftrightarrow$ 2:

$$(*) \qquad f_3^\epsilon(x^\epsilon) = \epsilon^3 f_3(x) \text{ and } g_3^\epsilon(x^\epsilon) = \epsilon^4 g_3(x).$$

1.4. Convergence Theorem

Under the assumptions that the Lamé's coefficients λ and μ are independent of ϵ and the applied forces satisfy $(*)$, P.-G. Ciarlet and P. Destuynder have shown [1], [2] that the vertical displacement $u_3(\epsilon)$ solution of the three-dimensional problem converges toward the vertical displacement $\zeta(\epsilon)$ solution to the two-dimensional problem (both of those displacements are expressed in the fixed set Ω),

$$\lim_{\epsilon\to 0} \|u_3(\epsilon) - \zeta(\epsilon)\|_{H^1(w)} \text{ „ } 0 .$$

2. Spectral Approximation

2.1. Motivation

A discretization method applied to the three-dimensional model will work well when the mesh size is substantially smaller than the thickness of the plate. While such a method cannot therefore be used for studying the displacement field for *a* very thin plate [5], our aim is to show that a *spectral method with respect to the thickness* gives good results.

2.2. Spectral Approximation

A spectral method [4] consists in seeking the solution to a boundary-value problem in terms of a truncated series of known, smooth functions (Pi). For example, the choice of polynomials as basis functions has been proven to be optimal in case of bending beams [7]. Let $\tilde{u}_N^\epsilon$ be the expansion of order N of the displacement u^ϵ:

$$\tilde{u}_N^\epsilon(x^\epsilon) = \sum_{i=0}^{N} u^{i,\epsilon}(x_1^\epsilon, x_2^\epsilon) P_i(x_3^\epsilon)$$

where

$$\begin{cases} u^{i,\epsilon} \text{ is only } x_1^\epsilon, \text{+-dependent } (x_3^\epsilon \text{ independent}) \\ P_i \text{ is only } x_3^\epsilon\text{-dependent function.} \end{cases}$$

Let us now present the Galerkin and Tau approximations. First the problem to be solved will be expressed in a more general form: the displacement u^ϵ is the solution in a certain space V^ϵ to the boundary-value problem:

$$\begin{cases} L^\epsilon u^\epsilon = f & \text{in } \Omega^\epsilon, \\ D^\epsilon u^\epsilon = g^{\epsilon\pm} & \text{on } \Gamma_\pm^\epsilon, \\ u^\epsilon = 0 & \text{on } \Gamma_0^\epsilon. \end{cases}$$

where L^ϵ and D^ϵ are linear partial differential operators and the associated *variational* formulation of this problem is the following: find $u_\epsilon \in V^\epsilon$ such that

$$B^\epsilon(u^\epsilon, v^\epsilon) = (f^\epsilon, v^\epsilon), \qquad \forall v^\epsilon \in V^\epsilon$$

where the bilinear quadratic form $B^\epsilon(u^\epsilon, v^\epsilon)$ is symmetric and elliptic.

The Galerkin approximation is constructed as follows: The Galerkin approximation $\tilde{u}_N^\epsilon$ of order N is the projection of the solution u^ϵ onto the space

$$V_N^\epsilon = \left\{ \sum_{i=0}^{N} v^{i,\epsilon} P_i \, , \; v^{i,\epsilon}\,|\,7 = 0 \, , \; i = 0 \, , \; N \right\} \subset V^\epsilon$$

with respect to the inner product associated to the quadratic form $B^\epsilon(u^\epsilon, v^\epsilon)$. Then, the expansion coefficients $(u^{i,\epsilon})_{0 \leq i \leq N}$ are solutions of the variational problem:

$$\sum_{i=0}^{N} B^\epsilon(u^{i,\epsilon} P_i, v^{j,\epsilon} P_j) = (f^\epsilon, v^{j,\epsilon} P_j), \quad \forall v^{j,\epsilon}|_\gamma = o \; , \quad j = 0, \ldots, N$$

where (,) is the inner product in $L^2(-\epsilon, +\epsilon)$.

The Tau approximation (introduced by Lanczos) is constructed as follows:

$$\tilde{u}_N^\epsilon(x^\epsilon) = \sum_{i=0}^{N+2} u^{i,\epsilon}(x_1^\epsilon, x_2^\epsilon) P_i(x_3^\epsilon) .$$

The $N + 3$ expansion coefficients $\{u^{i,\epsilon}\}$ are determined by the $N + 1$ equations:

$$\sum_{i=0}^{N+2} (L^\epsilon u^{i,\epsilon} P_i, P_j) = (f^\epsilon, P_j), \qquad 0 \leq j \leq N$$

and by the two boundary-conditions

$$\sum_{i=0}^{N+2} D^\epsilon u^{i,\epsilon} P_i(\pm\epsilon) = g^{\epsilon\pm} .$$

The boundary condition on Γ_0 implies that the expansion coefficients $(u^{i,\epsilon})_{0\leq i\leq N+2}$ vanish on the boundary 7.

Before giving the expression of the expansion coefficients, the operators L^ϵ and D^ϵ will be expressed in a more suitable form.

3. Another Expression of Equilibrium and Constitutive Equations – Variational Formulation

Let us introduce these new notations:

$\hat{u}$ is the vector of the first two components of any vector u.
$\hat{\sigma}^\epsilon$ is the vector whose components are σ_{13}^ϵ and σ_{23}^ϵ.
$\hat{\nabla}$ is the gradient with respect to the first two components $x_1^\epsilon, x_2^\epsilon$,

$$\hat{\nabla}^\epsilon u^\epsilon = \begin{pmatrix} \partial_1^\epsilon u^\epsilon \\ \partial_2^\epsilon u^\epsilon \end{pmatrix}.$$

$\hat{\Delta}^\epsilon$ is the Laplacian operator with respect to the first two components $x_1^\epsilon, x_2^\epsilon$.

$$\hat{\Delta}^\epsilon u^\epsilon = \partial_{11}^\epsilon u^\epsilon + \partial_{22}^\epsilon u^\epsilon .$$

The equilibrium equations on Ω^ϵ can be written as:

$$\begin{cases} f_\alpha^\epsilon = -(\mathrm{div}^\epsilon \sigma^\epsilon)_\alpha = -(\partial_1^\epsilon \sigma_{\alpha 1}^\epsilon + \partial_2^\epsilon \sigma_{\alpha 2}^\epsilon) - \partial_3^\epsilon \sigma_{\alpha 3}^\epsilon , \quad 1 \le \alpha \le 2 \\ \qquad = -\mu(\partial_{11}^\epsilon u_\alpha^\epsilon + \partial_{22}^\epsilon u_\alpha^\epsilon) - (\lambda + \mu)\partial_\alpha^\epsilon(\partial_1^\epsilon u_1^\epsilon + \partial_2^\epsilon u_2^\epsilon) - \lambda \partial_{\alpha 3}^\epsilon u_3^\epsilon - \partial_3^\epsilon \sigma_{\alpha 3} \\ f_3^\epsilon = -(\mathrm{div}^\epsilon \sigma^\epsilon)_3 = -(\partial_1^\epsilon \sigma_{31}^\epsilon + \partial_2 \sigma_{32}^\epsilon) - \partial_3^\epsilon \sigma_{33}^\epsilon \\ \qquad = -\mu(\partial_{11}^\epsilon u_3^\epsilon + \partial_{22}^\epsilon u_3^\epsilon) - \mu \partial_3^\epsilon(\partial_1^\epsilon u_1^\epsilon + \partial_2^\epsilon u_2^\epsilon) - \partial_3^\epsilon \sigma_{33}^\epsilon \end{cases}$$

or, in a more compact form as:

$$\begin{cases} -\mu \hat{\Delta}^\epsilon \hat{u}^\epsilon - (\lambda + \mu)\hat{\nabla}^\epsilon(\mathrm{div}^\epsilon \hat{u}^\epsilon) - \lambda \partial_3^\epsilon \hat{\nabla}^\epsilon u_3^\epsilon - \partial_3^\epsilon \hat{\sigma}^\epsilon = \hat{f}^\epsilon \quad \text{in } \Omega^\epsilon, \\ -\mu \hat{\Delta}^\epsilon u_3^\epsilon - \mu \partial_3^\epsilon(\mathrm{div}^\epsilon \hat{u}^\epsilon) - \partial_3^\epsilon \sigma_{33}^\epsilon = f_3^\epsilon \quad \text{in } \Omega^\epsilon. \end{cases}$$

And the boundary conditions on $\Gamma_\pm^\epsilon$ can be written as follows:

$$\begin{cases} \sigma_{\alpha 3}^\epsilon = \mu(\hat{\nabla}^\epsilon u_3^\epsilon + \partial_3^\epsilon \hat{u}^\epsilon) = \hat{g}^{\pm\epsilon} \quad \text{on } \Gamma_\pm^\epsilon, \\ \sigma_{33}^\epsilon = \lambda\, \mathrm{div}^\epsilon \hat{u}^\epsilon + (\lambda + 2\mu)\partial_3^\epsilon u_3^\epsilon = g_3^{\pm\epsilon} \quad \text{on } \Gamma_\pm^\epsilon. \end{cases}$$

or, in a more compact form as:

$$\begin{cases} \mu \hat{\nabla}^\epsilon u_3^\epsilon + \mu \partial_3^\epsilon \hat{u}^\epsilon = \hat{g}^{\epsilon\pm} \quad \text{on } \Gamma_\pm^\epsilon, \\ \lambda\, \mathrm{div}^\epsilon \hat{u}^\epsilon + (\lambda + 2\mu)\partial_3^\epsilon u_3^\epsilon = g_3^{\epsilon\pm} \quad \text{on } \Gamma_\pm^\epsilon . \end{cases}$$

From now on, we will drop the ^ and ϵ signs above A and V operators whenever no confusion shouldarise.

The problem to solve is then to find the horizontal ($\hat{u}^\epsilon$) and vertical (u_3^ϵ) components of the displacement such that:

$$(1^\epsilon) \qquad \begin{cases} -\mu \Delta \hat{u}^\epsilon - (\lambda + \mu)\nabla\, \mathrm{div}\, \hat{u}^\epsilon - \lambda \partial_3 \nabla u_3^\epsilon - \partial_3 \hat{\sigma}^\epsilon = \hat{f}^\epsilon \quad \text{in } \Omega^\epsilon, \\ -\mu \Delta u_3^\epsilon - \mu \partial_3\, \mathrm{div}\, \hat{u}^\epsilon - \partial_3 \sigma_{33}^\epsilon = f_3^\epsilon \quad \text{in } \Omega^\epsilon. \end{cases}$$

with the boundary conditions:

$$\begin{cases} \mu(\nabla u_3^\epsilon + \partial_3 \hat{u}^\epsilon) = \hat{g}^{\pm\epsilon} \quad \text{on } \Gamma_\pm^\epsilon, \\ \lambda \operatorname{div} \hat{u}^\epsilon + (\lambda + 2\mu)\partial_3 u_3^\epsilon = g_3^{\pm\epsilon} \quad \text{on } \Gamma_\pm^\epsilon, \\ \hat{v}^\epsilon = u_3^\epsilon = 0 \quad \text{on } \Gamma_0^\epsilon. \end{cases}$$

Let V^ϵ denote the separable Hilbert space $V^\epsilon = \{v^\epsilon \in \left(H^1(\Omega^\epsilon)\right)^3, \text{v} = 0 \text{ on } \Gamma_0^\epsilon\}$ equipped with the inner product $\left((u^\epsilon, v^\epsilon)\right) = \sum_{i=1}^3 \left((u_i^\epsilon, v_i^\epsilon)\right)_{H^1(\Omega^\epsilon)}$ and the norm $\|u^\epsilon\|_{V^\epsilon} = \left(\sum_{i=1}^3 \|u_i^\epsilon\|_{H^1(\Omega^\epsilon)}^2\right)^{1/2}$.

The variational formulation of (1') is:

$$(2^\epsilon) \qquad \begin{cases} u^\epsilon \in V^\epsilon, \\ B^\epsilon(u^\epsilon, v^\epsilon) = (f^\epsilon, v^\epsilon) + \int_{\Gamma_+^\epsilon \cup \Gamma_-^\epsilon} g^{\epsilon\pm} v^\epsilon, \quad \forall v^\epsilon \in V^\epsilon \end{cases}$$

where B^ϵ denotes the bilinear form:

$$B^\epsilon = \mu(\nabla \hat{u}, \nabla \hat{v}) + (\lambda + \mu)(\operatorname{div} \hat{u}, \operatorname{div} \hat{v}) - \lambda(\partial_3 \nabla u_3, \hat{v}) + \mu(\nabla u_3, \partial_3 \hat{v}) + \mu(\partial_3 \hat{u}, \partial_3 \hat{v})$$
$$+ \mu(\nabla u_3, \nabla v_3) - \lambda(\hat{u}, \partial_3 \nabla u_3) + \mu(\partial_3 \hat{u}, \nabla v_3) + (\lambda + 2\mu)(\partial_3 u_3, \partial_3 v_3)$$

and $(u, v) = \int_{\Omega^\epsilon} uv$.

It is easy to prove that B^ϵ is continuous on $V^\epsilon \times V^\epsilon$ and, using Korn inequality [3], that B^ϵ is coercive on V^ϵ, therefore the problem (2^ϵ) has a unique solution.

4. Galerkin Approximation

4.1. Convergence Theorem

In this section we shall follow [7] to give an estimate of $\|u^\epsilon - \tilde{u}_N^\epsilon\|$ where $\tilde{u}_N^\epsilon$ is the Galerkin approximation of order N of the solution u^ϵ of (2'). To simplify the computation we will assume that $f_3^\epsilon = g_3^{\epsilon-} = 0$, and the generalization becomes straightforward.

(a) Definition of V_N^ϵ

For any integer j_0, the following Neumann system of equations define the sequence of linearly independent elements of $(H^1(-1,1))^2, (Q_j^+, P_j^+)$ up to a constant in $(Q_{j_0}^+, P_{j_0}^+)$:

For any $-1 \leq j \leq j_0 - 1$, and any $z \in H^1(-1,+1)$,

$$\begin{cases} (\lambda + 2\mu) \int_{-1}^1 Q_{j+1}^{+'} z' + \mu \int_{-1}^1 P_j^{+'} z - \lambda \int_{-1}^1 P_j^+ z' + \mu \int_{-1}^1 Q_j^+ z = z(+1)\delta_j^1 \\ \mu \int_{-1}^1 P_{j+1}^{+'} z' - \lambda \int_{-1}^1 Q_{j+1}^{+'} z + \mu \int_{-1}^1 Q_{j+1}^+ z' + (\lambda + 2\mu) \int_{-1}^1 P_j^+ z = 0 \end{cases}$$

with $Q_{-1}^+, P_{-1}^+ = 0$.

We shall therefore define the approximation space $V_N^\epsilon \subset V^\epsilon$ as:

$$V_N^\epsilon = \left\{ \sum_{j=0}^N \epsilon^{2j} \begin{pmatrix} P_j^+(\epsilon x_3)\hat{v}^{\epsilon,j}(x_1^\epsilon, x_2^\epsilon) \\ Q_j^+(\epsilon x_3) v_3^{\epsilon,j}(x_1^\epsilon, x_2^\epsilon) \end{pmatrix}, \quad \hat{v}^{\epsilon,j} = v_3^{\epsilon,j} = 0, \text{ on } \gamma, 0 \leq j \leq N \right\}.$$

(b) Estimate of $\|u^\epsilon - \tilde{u}_N^\epsilon\|$

If the data $g_3^{\epsilon+}$ is smooth enough we can, for any $N \geq 0$, choose $w_N^\epsilon \in V_N^\epsilon$ as:

$$w_N^\epsilon = \sum_{j=0}^N \begin{pmatrix} P_j^+(\epsilon x_3)\nabla(-\Delta)^{j-2} g_3^{\epsilon+} \\ Q_j^+(\epsilon x_3)(-\Delta)^{j-2} g_3^{\epsilon+} \end{pmatrix}.$$

Because of the properties of the operator B^ϵ and of the definition of $\tilde{u}_N^\epsilon$ as the projection of u^ϵ onto VI;, it follows that there exists a constant C (independent of N and ϵ) such that:

$$\|u^\epsilon - \tilde{u}_N^\epsilon\|_{V^\epsilon} \leq C \inf_{w^\epsilon \in V^\epsilon} \|u^\epsilon - w^\epsilon\|_{V^\epsilon} \leq C \|u^\epsilon - w_N^\epsilon\|_{V^\epsilon} \leq C\, B^\epsilon(u^\epsilon - w_N^\epsilon, u^\epsilon - w_N^\epsilon)^{1/2}.$$

We shall compute $B^\epsilon(\,.,.)$ in the fixed set Ω. The variational formulation of (1') in the fixed set Ω is:

$$\text{(2)} \qquad \begin{cases} u \in V = \{v \in (H^1(\Omega))^3, v = 0 \text{ on } \Gamma_0\} \\ B(u,v) = \epsilon^4 \int_{\Gamma^+} g_3^+ v_3(+1), \qquad \forall v \in v \end{cases}$$

where

$$B(u,v) = \epsilon^4\mu(\nabla 3, \mathrm{V}\,\hat{v}) + \epsilon^4(\lambda+\mu)(\mathrm{div}\,\hat{u}, \mathrm{div}\,\hat{v}) - \epsilon^2\lambda(\partial_3\nabla u_3, \hat{v}) + \epsilon^2\mu(\nabla u_3, \partial_3\hat{v}) + \epsilon^2\mu(\partial_3\hat{u}, \partial_3\hat{v})$$
$$+ \epsilon^2\mu(\nabla u_3, \nabla v_3) - \lambda\epsilon^2(\hat{u}, \partial_3\nabla v_3) + \epsilon^2\mu(\partial_3\hat{u}, \nabla v_3) + (\lambda + 2\mu)(\partial_3 u_3, \partial_3 v_3)$$

and (u, v) = $\int_\Omega uv$.

Therefore, for any $z \in \left(H^1(\Omega)\right)^3$ we get that:

$$B(u - w_N, z) = \epsilon^4 \int_{\Gamma+} g_3^+ z(+1) - B(w_N, z)$$

and because of the property of w_N,

$$B(u - w_N, z) = \epsilon^{2N+4}\{\mu(P_{N+1}^{+'}\hat{t}, \hat{z}') - \lambda(Q_{N+1}^{+'}\hat{t}, \hat{z}) + \mu(Q_{N+1}^{+}\hat{t}, \hat{z}')\} + \epsilon^{2N+2}(\lambda + 2\mu)(Q_{N+1}^{+'}t_3, z_3')$$

with $t_3 = (-\Delta)^{N-1}g_3^+$, $\hat{t} = \nabla t_3$.

Then, there exists a constant C_N independent of ϵ such that:

$$|B(u - w_N, z)| \leq C_N \epsilon^{2N+2} \|z\|_V$$

and the final estimate is obtained as:

$$\|u^\epsilon - \tilde{u}_N^\epsilon\|_{V^\epsilon} \leq C\, |B^\epsilon(u^\epsilon - w_N^\epsilon, u^\epsilon - w_N^\epsilon)| \leq C C_N \epsilon^{2N+(3/2)}\,.$$

4.2. Galerkin Equations

The Galerkin expansion coefficients u_i^ϵ are given by the following equations:

$$\begin{cases} -\mu \sum_{i=1}^{N} \Delta\hat{u}^{\epsilon,i}(P_i, Pj) - (\lambda + \mu)\sum_{i=1}^{N} \nabla \operatorname{div} \hat{u}^{\epsilon,i}(P_i, Pj) - \lambda \sum_{i=1}^{N} \nabla u_3^{\epsilon,i}(P_i', Pj) - (\partial_3\hat{\sigma}^\epsilon, Pj) \\ \qquad = (\hat{f}^\epsilon, P_j) \qquad \text{on } w,\ 0 \leq j \leq N \\ -\mu \sum_{i=1}^{N} \Delta u_3^{\epsilon,i}(P_i, P_j) - \mu \sum_{i=1}^{N} \operatorname{div} \hat{u}^{\epsilon,i}(P_i', P_j) - (\partial_3\sigma_{33}^\epsilon, Pj) = (f_3^\epsilon, Pj)\,, \quad \text{on } w,\ 0 \leq j \leq N \end{cases}$$

and $u^{\epsilon,i} = 0$ on $7, 0 \leq i \leq N$:

An integration by parts gives for any $1 \leq j \leq N$:

$$\begin{aligned} -(\partial_3\hat{\sigma}^\epsilon, P_j) &= (\hat{\sigma}^\epsilon, P_j') - [\hat{g}^{\epsilon+}P_j(\epsilon) - \hat{g}^{\epsilon-}P_j(-\epsilon)] \\ &= \mu \sum_{i=1}^{N} \nabla u_3^{\epsilon,i}(P_i, P_j') + \mu \sum_{i=1}^{N} \hat{u}^{\epsilon,i}(P_i', P_j') - [\hat{g}^{\epsilon+}P_j(\epsilon) - \hat{g}^{\epsilon-}P_j(-\epsilon)] \end{aligned}$$

and

$$-(\partial_3\sigma_{33}^\epsilon, Pj) = \lambda \sum_{i=1}^{N} \operatorname{div} \hat{u}^{\epsilon,i}(P_i, P_j') + (\lambda + 2\mu)\sum_{i=1}^{N} u_3^{\epsilon,i}(P_i', P_j') - [g_3^{\epsilon+}P_j(\epsilon) - g_3^{\epsilon-}P_j(-\epsilon)]\,.$$

Thus, the Galerkin equations become

$$
\begin{cases}
-\mu \sum_{i=0}^{N} \Delta \hat{u}^{\epsilon,i}(P_i, P_j) - (\lambda+\mu) \sum_{i=0}^{N} \nabla \operatorname{div} \hat{u}^{\epsilon,i}(P_i, P_j) + (\lambda+\mu) \sum_{i=0}^{N} \nabla u_3^{\epsilon,i}(P_i, P_j') \\
\quad + \mu \sum_{i=0}^{N} \hat{u}^{\epsilon,i}(P_i', P_j') - \lambda \sum_{i=0}^{N} \nabla u_3^{\epsilon,i}[P_i P_j]_{-\epsilon}^{\epsilon} \\
\quad = (\hat{f}^{\epsilon}, P_j) + [\hat{g}^{\epsilon+} P_j(\epsilon) - \hat{g}^{\epsilon-} P_j(-\epsilon)], \quad \text{on } w, 0 \le j \le N \\
-\mu \sum_{i=0}^{N} \Delta u_3^{\epsilon,i}(P_i, P_j) + (\lambda+\mu) \sum_{i=0}^{N} \operatorname{div} \hat{u}^{\epsilon,i}(P_i, P_j') + (\lambda+2\mu) \sum_{i=0}^{N} u_3^{\epsilon,i}(P_i', P_j') \\
\quad - \mu \sum_{i=0}^{N} \operatorname{div} \hat{u}^{\epsilon,i}[P_i P_j]_{-\epsilon}^{\epsilon} \\
\quad = (f_3^{\epsilon}, P_j) + [g_3^{\epsilon+} P_j(\epsilon) - g_3^{\epsilon-} P_j(-\epsilon)], \quad \text{on } w, 0 \le j \le N
\end{cases}
$$

and $u_3^{\epsilon,i} = \hat{u}^{\epsilon,i} = 0$ on γ, $0 \le i \le N$.

4.3. Choice of Approximation Functions

For computational purposes we chose *polynomials* as approximation functions *(Pi):*

$$P_0(t) = 1$$

$$P_1(t) = \frac{t}{\epsilon}$$

$$Pj(t) = \int_{-\epsilon}^{t} L_j(t)\,dt$$ where L_j is the j^th^ Legendre polynomial on $(-\epsilon, +\epsilon)$, $j \ge 2$.

The matrix (P_i, P_j) *is* ***pentadiagonal***;
for example, the matrix of order 5 is

$$
2\epsilon \begin{pmatrix}
1 & 0 & -1/3 & 0 & 0 \\
 & & & & \\
0 & -1/15 & 0 & 2/105 & 0 \\
-0/3 & 1/0 & -1/105 & -10/15 & 2/315
\end{pmatrix}
$$

The matrix (P_i, P_j') *is* ***tridiagonal***;
for example, the matrix of order 5 is

$$
2 \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & +1/3 & 0 & 0 \\
0 & -1/3 & 0 & +1/15 & 0 \\
0 & 0 & -1/15 & 0 & +1/35 \\
0 & 0 & 0 & -1/35 & 0
\end{pmatrix}
$$

The matrix (P_i', P_j') *is diagonal;*

and the diagonal elements are of the form $\frac{2}{\epsilon}\left(\frac{1}{2i-1}\right)$ **with** $(P_0', P_0') = 0$.

The values of $P_i(\pm\epsilon)$ **and** *of* $P_i'(\pm\epsilon)$ **are given by:**
$P_0(\pm\epsilon) = 1$, $P_1(\pm\epsilon) = \pm 1$, $P_i(\pm\epsilon) = 0$, $i \geq 2$, $P_{2k+1}(0) = 0$, $k \geq 0$.

The quantities $[P_i P_j]_{-\epsilon}^{\epsilon} = P_i(\epsilon)P_j(\epsilon) - P_i(-\epsilon)P_j(-\epsilon)$ **vanish for** i **or** $j \geq 2$, **and** $[P_0 P_0]_{-\epsilon}^{\epsilon} = 0$, $[P_1 P_1]_{-\epsilon}^{\epsilon} = 0$, $[P_0 P_1]_{-\epsilon}^{\epsilon} = [P_1 P_0]_{-\epsilon}^{\epsilon} = 2$.

Remark. With this choice of polynomials we observe that the $3(N+1)$ **equations split into 2 sets of equations; the first one with the unknowns** $\{\hat{u}^{\epsilon,2k}, u_3^{\epsilon,2k+1}\}$ **and the second one with the unknowns** $\{\hat{u}^{\epsilon,2k+2}, u_3^{\epsilon,2k}\}$.

Since $P_{2k+1}(0) = 0$, **the displacement in the middle surface** $x_3 = 0$ **depends only on the even coefficients** $u^{\epsilon,2k}$. **Because we are primarily interested in the vertical displacement on the middle surface, we will use the second set of equations (giving** $\hat{u}^{\epsilon,2k+1}$ **and** $u_3^{\epsilon,2k}$**).**

To simplify the study of the convergence when the thickness goes to zero we shall rewrite the Galerkin equations in the fixed set Ω **with the** *following* **notations:**

$$\tilde{u}_{N,\alpha}^{\epsilon}(x^{\epsilon}) = \epsilon^2 \hat{u}_{N,\alpha}(\epsilon, x) = \epsilon^2 \sum_{i=0}^{N} (\epsilon^2)^{\mathrm{int}(i/2)} \hat{v}^i P_i = \epsilon^2 \{v_\alpha^0 P_0 + v_\alpha^1 P_1 + \epsilon^2 v_\alpha^2 P_2 + \epsilon^2 v_\alpha^3 P_3 + \cdots\}$$

$$\tilde{u}_{N,3}^{\epsilon}(x^{\epsilon}) = \epsilon u_{N,3}(\epsilon, x) = \epsilon \sum_{i=0}^{N} (\epsilon^2)^{\mathrm{int}(i/2)} v_3^i P_i = \epsilon\{v_3^0 P_0 v_3^1 P_1 \epsilon^2 v_3^2 P_2 \epsilon^2 v_3^3 P_3$$

where int (z) **is the integer part of** z **and**

$$F_3^{\epsilon,j} = (f_3^{\epsilon}, P_j) + [g_3^{\epsilon+} P_j(\epsilon) - g_3^{\epsilon-} P_j(-\epsilon)] = \epsilon^4 F_3^j.$$

4.4. Galerkin Approximation of Order 2: The Direct System

In the fixed set Ω **the displacement of order 2 is sought in the following form:**

$$\tilde{u}_2(\epsilon, x) = v^0 + v^1 x_3 + \epsilon^2 v^2 \frac{x_3^2 - 1}{2}$$

and therefore the vertical displacement on the middle surface $x_3 = 0$ **is equal to** $v_3^0 - \epsilon^2 \frac{v_3^2}{2}$.

The expansion coefficients $\hat{v}^1, v_3^0, v_3^2$ **are solutions to the homogeneous Dirichlet problem:**

$$\text{(I)} \quad \begin{cases} -\mu \frac{\epsilon^2}{3} \Delta \hat{v}^1 - (\lambda + \mu) \frac{\epsilon^2}{3} \nabla \operatorname{div} \hat{v}^1 + \mu \nabla v_3^0 - (\lambda + \mu) \frac{\epsilon^2}{3} \nabla v_3^2 + \mu \hat{v}^1 = 0 & \text{in } \omega, \\ \Delta v_3^0 - \frac{\epsilon^2}{3} \Delta v_3^2 + \operatorname{div} \hat{v}^1 = -\frac{\epsilon^2}{2\mu} F_3^0 & \text{in } \omega, \\ \frac{\mu}{3} \Delta v_3^0 - \frac{2\mu}{15} \epsilon^2 \Delta v_3^2 + \frac{(\lambda+\mu)}{3} \operatorname{div} \hat{v}^1 + \frac{\lambda+2\mu}{3} v_3^2 = \frac{\epsilon^2 F_3^2}{2} & \text{in } \omega. \end{cases}$$

and $\hat{v}^1 = v_3^0 = v_3^2 = 0$ **on** γ.

In this section we shall study the existence and uniqueness of the solution to this system.

(a) Existence and Uniqueness of the Solution.

The system (I) can be written in the following equivalent form:

$$\text{(II)}\quad \begin{cases} -\mu \frac{\epsilon^2}{3}\Delta\hat{v}^1 - (\lambda+\mu)\frac{\epsilon^2}{3}\nabla \operatorname{div}\hat{v}^1 + \mu\nabla v_3^0 - (\lambda+\mu)\frac{\epsilon^2}{3}\nabla u_3^2 + \mu\hat{v}^1 = 0 & \text{in w},\\ -\frac{\mu}{15}\Delta v_3^0 + \frac{5\lambda-\mu}{15}\operatorname{div}\hat{v}^1 + \frac{\lambda+2\mu}{3}v_3^2 = \epsilon^2\frac{F_3^2}{2} + \epsilon^2\frac{F_3^0}{5} & \text{in } \omega,\\ -\frac{\epsilon^2}{15}\mu\Delta v_3^2 + \lambda\operatorname{div}v^1 + (\lambda+2\mu)v_3^2 = \frac{3\epsilon^2 F_3^2}{2} + \frac{\epsilon^2}{2}F_3^0 & \text{in } \omega. \end{cases}$$

and $\hat{v}^1 = v_3^0 = v_3^2 = 0$ on γ.

Let V denote the Sobolev space $V = \{v = (\hat{v}^1, v_3^0, v_3^2),\ v \in (H_0^1(w))^4\}$ equipped with the usual inner product, and A denote the bilinear form associated with the variational formulation of system (II). Then A is continuous on $V \times V$ and satisfies Gärding inequality [5], $A(v,v) \geq \alpha|\nabla v|^2 - \beta|v|^2$ where α and β are positive constants and where $|\,|$ is the norm $(L^2(w))^4$. Therefore, if 0 is not an eigenvalue of the previous system, we can conclude that this system has a unique solution in V.

(b) Asymptotic Behavior of the Solution.

The system (I) is singular when $\epsilon = 0$. (The second equation is the divergence of the first one.) In order to avoid this singularity we eliminate the horizontal component $\hat{v}^1$ in two equations and we obtain the following system (III) of higher order:

$$\text{(III)}\quad \begin{cases} -\mu \frac{\epsilon^2}{3}\Delta\hat{v}^1 - (\lambda+\mu)\frac{\epsilon^2}{3}\nabla \operatorname{div}\hat{v}^1 + \mu\nabla v_3^0 - (\lambda+\mu)\frac{\epsilon^2}{3}\nabla v_3^2 + \mu\hat{v}^1 = 0 & \text{in } \omega,\\ \Delta v_3^0 - \epsilon^2\Delta v_3^2\left\{\frac{1}{3} - \frac{\mu}{15\lambda}\right\} - \frac{\lambda+2\mu}{\lambda}v_3^2 = -\frac{3\epsilon^2}{2\lambda}F_3^2 - \frac{\lambda+\mu}{2\lambda\mu}\epsilon^2 F_3^0 & \text{in } \omega,\\ -\frac{\lambda+2\mu}{15\lambda}\mu\epsilon^2\Delta^2 v_3^2 + \frac{4\mu(\lambda+\mu)}{\lambda}\Delta v_3^2 = \frac{3(\lambda+2\mu)}{2\lambda}\epsilon^2\Delta F_3^2 + \frac{\lambda+2\mu}{2\lambda}\epsilon^2\Delta F_3^0 + \frac{3F_3^0}{2} & \text{in } \omega. \end{cases}$$

with boundary conditions

$$\begin{cases} \hat{v}^1 = v_3^2 = v_3^0 = 0 & \text{on } \gamma,\\ \operatorname{div}\hat{v}^1 + \Delta v_3^0 - \epsilon^2\Delta v_3^2 = -\frac{\epsilon^2}{2\mu}F_3^0 & \text{on}\gamma. \end{cases}$$

When $\epsilon = 0$ this system has a unique solution $\hat{v}^{1*}, v_3^{0*}, v_3^{2*}$ such that:

$$\begin{cases} \Delta^2 v_3^{0*} = \frac{3}{2}(1-\nu^2)F_3^0 & \text{in w},\\ \Delta v_3^{0*} = \frac{\lambda+2\mu}{\lambda}v_3^{2*} & \text{in } \omega,\\ \hat{v}^{1*} = -\nabla v_3^{0*} & \text{in w}. \end{cases}$$

and the boundary conditions are obtained by continuity:

$$\begin{cases} v_3^{0*} = \frac{\partial v_3^{0*}}{\partial n} = 0 & \text{on } \gamma,\\ \hat{v}^{1*} = \hat{v}_3^{2*} = 0 & \text{on}\gamma. \end{cases}$$

Then, v_3^{0*} is the solution to the two-dimensional problem and we observe that the condition $\Delta v_3^{0*} = \frac{\lambda+2\mu}{\lambda}v_3^{2*} = 0$ on γ introduces boundary layers.

In the next section we will study the modified system (III).

4.5. Galerkin Approximation of Order 2: The Modified System

(a) Existence and Uniqueness of the Solution.

We can rearrange system (III) as:

If $\overline{v}_3 = v_3^0 - \epsilon^2 v_3^2$, the vector $(\hat{v}_1\overline{v}_3, v_3^2)$ is solution to the system (IV),

$$\text{(IV)} \qquad \begin{cases} -\mu \frac{\epsilon^2}{3} \Delta \hat{v}^1 - (\lambda+\mu)\frac{\epsilon^3}{3} \nabla \operatorname{div} \hat{v}^1 + \mu \nabla \overline{v}_3 - \frac{\lambda\epsilon^2}{3} \nabla v_3^2 + \mu \hat{v}^1 = 0 & \text{in } \omega, \\ \Delta^2 \overline{v}_3 - \frac{\lambda}{\lambda+2\mu} \Delta v_3^2 = \frac{3F_3^0}{2(\lambda+2\mu)} - \frac{\epsilon^2 \Delta F_3^0}{2\mu} & \text{in w,} \\ -\frac{\epsilon^2\mu}{15\lambda} \Delta v_3^2 + \frac{\lambda+2\mu}{\lambda} v_3^2 - \Delta \overline{v}_3 = \frac{3\epsilon^2}{2\lambda} F_3^2 + \frac{\lambda+\mu}{2\lambda\mu} \epsilon^2 F_3^0 & \text{in w.} \end{cases}$$

with the boundary conditions

$$\begin{cases} \hat{v}^1 = \overline{v}_3 = v_3^2 = 0 & \text{on } \gamma, \\ \operatorname{div} \hat{v}^1 + \Delta \overline{v}_3 = -\frac{\epsilon^2}{2\mu} F_3^0 & \text{on } \gamma. \end{cases}$$

Let V denote the space

$$V = \{v = (\hat{v}^1, \overline{v}_3, v_3^2),\ \hat{v}_1 \in \left(H_0^1(w)\right)^2,\ \overline{v}_3 \in L^2(w), \Delta\overline{v}_3 \in L^2(w), \overline{v}_3 = 0 \text{ on } \gamma, v_3^2 H_0^1(w)\}$$

equipped with the inner product

$$(u,v)_V = (\hat{u}_1, \hat{v}_1)_{L^2(w)} + (\overline{u}_3, \overline{v}_3)_{L^2(w)} + (\Delta\overline{u}_3, \Delta\overline{v}_3)_{L^2(w)} + (u_3^2, v_3^2)_{H^1(w)}$$

and A denote the bilinear form associated with the variational formulation of system (IV). Then A is continuous on $V \times V$, and satisfies Gärding inequality:

$$A\left((\hat{v}^1, \overline{v}_3, v_3^0), (\hat{v}^1, \overline{v}_3, v_3^0)\right) \geq \alpha(|\nabla \hat{v}^1|^2 + |\Delta \overline{v}_3|^2 + |\nabla v_3^2|^2) - \beta(|\hat{v}^1|^2 + |\overline{v}_3|^2 + |v_3^2|^2)$$

and $A(v, v) \geq \alpha |v|_V^2 - \beta |v|^2$, where α and β are positive constants and where $|\,|$ is the norm in $L^2(w)$. Therefore if 0 is not an eigenvalue of system (IV) we can conclude that this system has a unique solution in V, interpreting *formally* the boundary conditions.

(b) Computation of the Solution

System (IV) can be solved by an iterative "fixed-point" algorithm:

1. Computation of $\overline{v}_3^{(k+1)}$ as the solution of the bihar/ihar monic equation:

$$\Delta^2 \overline{v}_3^{(k+1)} = \frac{\lambda}{\lambda+2\mu} \Delta v_3^{2^{(k)}} + \frac{3F_3^0}{2(\lambda+2\mu)} - \epsilon^2 \frac{\Delta F_3^0}{2\mu} \quad \text{in } w,$$

$$\begin{cases} \overline{v}_3^{(k+1)} = 0 & \text{on } \gamma, \\ \Delta \overline{v}_3^{(k+1)} = -\operatorname{div} \hat{v}^{1^{(k)}} - \frac{\epsilon^2}{2\mu} F_3^0 & \text{on } \gamma. \end{cases}$$

According to the special type of boundary conditions this computation can be done simply by the use of a Poisson solver.

2. Computation of $v_3^{2^{(k+1)}}$ as the solution to the Laplace equation:

$$\begin{cases} -\frac{\epsilon^2\mu}{15\lambda} \Delta v_3^{2^{(k+1)}} + \frac{\lambda+2\mu}{\lambda} v_3^{2^{(k+1)}} = -\Delta \overline{v}_3^{(k+1)} + \frac{3\epsilon^2}{2\lambda} F_3^2 + \frac{\lambda+\mu}{2\lambda\mu} \epsilon^3 F_3^0 & \text{in } \omega, \\ v_3^{2^{(k+1)}} = 0 & \text{on 7.} \end{cases}$$

3. Computation of $\hat{v}_1^{(k+1)}$ as the solution to the Laplace system:

$$\begin{cases} -\mu \frac{\epsilon^2}{3} \Delta \hat{v}^{1^{(k+1)}} - (\lambda+\mu)\frac{\epsilon^2}{3} \nabla \operatorname{div} \hat{v}^{1^{(k+1)}} + \mu \hat{v}^{1^{(k+1)}} = -\mu \nabla \overline{v}_3^{(k+1)} - \frac{\lambda\epsilon^2}{3} \nabla v_3^{2^{(k+1)}} & \text{in } \omega, \\ \hat{v}^{1^{(k+1)}} = 0 & \text{on } \gamma. \end{cases}$$

5. Tau Approximation

5.1. Tau Equations

The Tau expansion coefficients are given by two kinds of equations: The first set of equations is of the same kind as the one obtained by Galerkin approximation:

$$\begin{cases} -\mu \sum_{i=0}^{N+2} \Delta \hat{u}^{\epsilon,i}(P_i, Pj) - (A + \mu) \sum_{i=0}^{N+2} \text{V div}\, u^{\epsilon,i}(P_i, Pj) - (\lambda + \mu) \sum_{i=0}^{N+2} \nabla \hat{u}_3^{\epsilon,i}(P_i', Pj) \\ \qquad - \mu \sum_{i=0}^{N+2} \hat{u}^{\epsilon,i}(P_i'', Pj) = (\hat{f}^\epsilon, Pj), \quad 0 \leq j \leq N \\ -\mu \sum_{i=0}^{N+2} \Delta u_3^{\epsilon,i}(P_i, Pj) - (A + \mu) \sum_{i=0}^{N+2} \text{div}\, \hat{u}^{\epsilon,i}(P_i', Pj) - (\lambda + 2\mu) \sum_{i=0}^{N+2} u_3^{\epsilon,i}(P_i'', Pj) \\ \qquad = (f_3^\epsilon, P_j), \quad 0 \leq j \leq N. \end{cases}$$

The second set is given by the boundary conditions:

$$\begin{cases} \mu \sum_{i=0}^{N+2} \nabla u_3^{\epsilon,i} P_i(\pm\epsilon) + \mu \sum_{i=0}^{N+2} \hat{u}^{\epsilon,i} P_i'(\pm\epsilon) = \hat{g}^{\epsilon\pm} \\ \lambda \sum_{i=0}^{N+2} \text{div}\, \hat{u}^{\epsilon,i} P_i(\pm\epsilon) + (A + 2\mu) \sum_{i=0}^{N+2} u_3^{\epsilon,i} \; P_i'(\pm\epsilon) = g_3^{\epsilon\pm} \end{cases}$$

and the coefficients $\hat{u}^{\epsilon,i}$ and $u_3^{\epsilon,i}$ vanish on the boundary 7.

5.2. Choice of Approximation Functions

In the case of Tau approximation we used Legendre polynomials as approximation functions (P_i).

We have also the following useful properties:

The matrix (P_i, P_j) *is* diagonal;

and the diagonal elements are of the form:

$$2\epsilon \left(\frac{1}{2i+1} \right) .$$

The matrix (P_i', P_j) is lower triangular;

for example, the matrix of order 5 is:

$$2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} .$$

The **matrix** (P_i'', P_j) **is** *lower* **tria lar;**
for example, the matrix of order 5 is:

$\frac{2}{\epsilon}$

$\epsilon 2$ 0 0 0 0 0 0000. 05000 00000

10 0 7 0 0

The values of $P_i(\pm\epsilon)$ and of $P_i'(\pm\epsilon)$ are given by:

$$
\begin{cases}
P_i(+\epsilon) = 1\,, \quad P_i(-\epsilon) = (-1)^i\,, & i \geq 0 \\
P_0' = 0\,, \quad P_i'(\pm\epsilon) = \pm(-1)^{i-1}\frac{i(i+1)}{2}\,, & i \geq 1\,.
\end{cases}
$$

The same remark and notations as for the Galerkin approximation can be done and used.

5.3. Tau Approximation of Order 2

In the fixed set Ω the displacement of order 2 is sought under the following form:

$$
\tilde{u}_2(\epsilon, x) = v^0 + v^1 x_3 + \epsilon^2 v^2 \,\frac{3x3 - 1}{2}
$$

and the vertical displacement on the middle surface $x_3 = 0$ is then equal to $v_3^0 - \epsilon^2 \frac{v_3^2}{2}$. The expansion coefficients $\hat{v}^1$, v_3^0, v_3^2 are solutions to the homogeneous Dirichlet problem:

$$
\begin{cases}
-\mu\Delta v_3^0 - (A + \mu)\,\mathrm{div}\,\hat{v}^1 - 3(\lambda + 2\mu)v_3^2 = \frac{\epsilon^2}{2} f_3^0 & \text{in w,} \\
\nabla v_3^0 + \epsilon^2 \nabla v_3^2 + \hat{v}^1 = 0 & \text{in w,} \\
\lambda\,\mathrm{div}\,\hat{v}^1 + 3(\lambda + 2\mu)v_3^2 = \epsilon^2 \frac{g_3^+ - g_3^-}{2}. & \text{in } \omega.
\end{cases}
$$

and $\hat{v}_1 = v_3^0 = v_3^2 = 0$ on γ.

For $\epsilon = 0$ the solution to this system does not verify the equations of the two-dimensional clamped plate. We shall then examine the approximation of order 3.

5.4. Tau Approximation of Order 3: The Complete System

In the fixed set Ω the displacemint of order 3 is sought under the following form:

$$
\tilde{u}_3(\epsilon, x) = v^0 + v^1 x_3 + \epsilon^2 v^2 \,\frac{3x_3^2 - 1}{2} + \epsilon^2 v^3 \,\frac{5x_3^3 - 3x_3}{2}\,.
$$

Therefore the vertical displacement on the middle surface $x_3 = 0$ is equal to $v_3^0 - \epsilon^2 \frac{v_3^2}{2}$. The expansion coefficients $\hat{v}^1$, $\hat{v}^3$, v_3^0, v_3^2 are solutions to the homogeneous Dirichlet problem:

(I)
$$
\begin{cases}
-\mu\Delta v_3^0 - (\lambda + \mu)(\mathrm{div}\,\hat{v}^1 + \epsilon^2\,\mathrm{div}\,\hat{v}^3) - 3(\lambda + 2\mu)v_3^2 = \frac{\epsilon^2 F_3^0}{2} & \text{in w,} \\
-\frac{\mu}{3}\,\Delta\hat{v}^1 - \frac{\lambda+\mu}{3}\,\nabla\,\mathrm{div}\,\hat{v}^1 - (\lambda + \mu)\nabla v_3^2 - 5\mu\,\hat{v}^3 = 0 & \text{in w,} \\
\mu(\nabla v_3^0 + \epsilon^2 \nabla v_3^2) + \mu\hat{v}^1 = 0 & \text{in w,} \\
\lambda(\mathrm{div}\,\hat{v}^1 + \epsilon^2\,\mathrm{div}\,\hat{v}^3) + 3(X + 2\mu)v_3^2 = 0 & \text{in } \omega.
\end{cases}
$$

and $\hat{v}^1 = \hat{v}^3 = v_3^0 = v_3^2$ on 7.

From the structure of system (I) we see that we can express $\hat{v}^1$ and $\hat{v}^3$ directly in terms of v_3^0 and v_3^2, making this substitution we therefore obtain the following reduced system (II).

$$\text{(II)} \qquad \begin{cases} \frac{2\epsilon^2}{15}(\lambda+2\mu)\Delta^2 v_3^2 - \frac{8(\lambda+\mu)}{\lambda}\mu\Delta v_3^2 = -F_3^0 + \epsilon^2\frac{\lambda+2\mu}{15\mu}\Delta F_3^0 & \text{in } \omega, \\ \Delta v_3^0 - \frac{3(\lambda+2\mu)}{\lambda}v_3^2 = -\frac{\epsilon^2}{2\mu}F_3^0, & \text{in } \omega. \end{cases}$$

and assuming that the third equation of system (I) can be continued on the boundary we get the necessary boundary conditions:

$$\begin{cases} \frac{\partial}{\partial n}v_3^0 + \epsilon^2\frac{\partial}{\partial n}v_3^2 = 0 & \text{on } \gamma, \\ v_3^0 = v_3^2 = 0 & \text{on } 7. \end{cases}$$

The values of $\hat{v}^1$ and $\hat{v}^3$ are then obtained using the second and third equations of system (I). But in this case the boundary condition $\hat{v}_3 = 0$ on 7 is satisfied only on the middle surface.

The same remarks as for Galerkin approximation can be done: the solution v_3^{0*}, v_3^{2*} of system (II) is such that:

$$\begin{cases} \Delta^2 v_3^{0*} = 3\frac{\lambda+2\mu}{\lambda}\Delta v_3^{2*} = \frac{3}{8}\frac{(\lambda+2\mu)}{(\lambda+\mu)\mu}F_3^0 = \frac{3}{2}(1-\nu^2)F_3^0 & \text{in } \omega, \\ v_3^{0*} = \frac{\partial}{\partial n}v_3^{0*} = 0 & \text{on } 7. \end{cases}$$

Therefore v_3^{0*} is a solution to the two-dimensional problem and we observe that the condition $\Delta v_3^{0*} = \frac{3(\lambda+2\mu)}{\lambda}v_3^{2*} = 0$ on 7 introduces boundary layers.

The next section will study the reduced system (II).

5.5. Tau Approximation of Order 3: The Reduced System

(a) Existence and Uniqueness of the Solution.

We can rewrite the system (II) in the following form:

If $\overline{v}_3 = v_3^0 + \epsilon^2 v_3^2$, the vector $(\overline{v}_3, v_3^2)$ is solution to the system (III).

$$\text{(III)} \qquad \begin{cases} \frac{2(\lambda+2\mu)}{15}\Delta^2\overline{v}_3 - \left(\frac{2}{5}\frac{(\lambda+2\mu)^2}{\lambda} + \frac{8(\lambda+\mu)\mu}{\lambda}\right)\Delta v_3^2 = -F_3^0 & \text{in } \omega, \\ -\mu\Delta\overline{v}_3 + \epsilon^2\mu\Delta v_3^2 + \frac{3(\lambda+2\mu)}{\lambda}\mu v_3^2 = +\frac{\epsilon^2}{2}F_3^0 & \text{in } \omega. \end{cases}$$

and

$$\begin{cases} \overline{v}_3 = \frac{\partial\overline{v}_3}{\partial n} = 0 & \text{on } \gamma, \\ v_3^2 = 0 & \text{on } 7. \end{cases}$$

Let $V = [H_0^2(w)] \times [\mathrm{H},{(w)}]$, and A the bilinear form associated with the variational formulation of system (III). Then A is continuous on $V \times V$ and satisfies the Gärding inequality:

$$A\big((\overline{v}_3, v_3^2),(\overline{v}_3, v_3^2)\big) \geq \alpha(|\Delta\overline{v}_3|^2 + |\nabla v_3^2|^2) - \beta|v_3^2|^2\,,$$

where α and β are positive constants, then $(Az,z) \geq \alpha\|z\|_V^2 - \beta|z|^2$. Therefore if 0 is not an eigenvalue of system (III), we can conclude that this system has a unique solution $\overline{v}_3 \in H_0^2(w)$, $v_3^2 \in H_0^1(w)$.

(b) Computation of the Solution.

System (III) is written in a more appropriate form for the computation:if $\bar{v}_3 = v_3^0 - \epsilon^2 \frac{(\lambda+2\mu)^2}{20(\lambda+\mu)\mu} v_3^2$,then the vector $(\bar{v}_3, v_3^2)$is solution to the system:

$$\begin{cases} \Delta^2 \bar{v}_3 = \frac{3(\lambda+2\mu)}{8(\lambda+\mu)\mu} F_3^0 - \frac{\epsilon^2}{2\mu}(1 + \frac{(\lambda+2\mu)^2}{20(\lambda+\mu)\mu})\Delta F_3^0 & \text{in w,} \\ -\epsilon^2 \frac{(\lambda+2\mu)^2}{20(\lambda+\mu)\mu}\Delta v_3^2 + 3\frac{(\lambda+2\mu)}{\lambda} v_3^2 - \Delta \bar{v}_3 = \frac{\epsilon^2}{2\mu} F_3^0 & \text{in w.} \end{cases}$$

with the boundary conditions:

$$\begin{cases} \bar{v}_3 = v_3^2 = 0 & \text{on } \gamma, \\ \frac{\partial}{\partial n}\bar{v}_3 + \epsilon^2(1 + \frac{(\lambda+2\mu)^2}{20(\lambda+\mu)\mu})\frac{\partial}{\partial n} v_3^2 = 0 & \text{on } \gamma. \end{cases}$$

This system is then computed using a "fixed-point" algorithm:

1. Computation of $\bar{v}_3^{k+1}$ as the solution to the biharmonic equation:

$$\begin{cases} \Delta^2 \bar{v}_3^{(k+1)} = \frac{3(\lambda+2\mu)}{8(\lambda+\mu)\mu} F_3^0 - \frac{\epsilon^2}{2\mu}(1 + \frac{(\lambda+2\mu)^2}{20(\lambda+\mu)\mu})\Delta F_3^0 & \text{in w,} \\ \bar{v}_3^{(k+1)} = 0 & \text{on } \gamma, \\ \frac{\partial}{\partial n}\bar{v}_3^{(k+1)} = -\epsilon^2(1 + \frac{(\lambda+2\mu)^2}{20(\lambda+\mu)\mu})\frac{\partial}{\partial n} v_3^{2(k)} & \text{on 7.} \end{cases}$$

2. Computation of $v_3^{2(k+1)}$ as the solution to Laplace equation

$$\begin{cases} -\epsilon^2 \frac{3(\lambda+2\mu)}{8(\lambda+\mu)\mu}\Delta v_3^{2(k+1)} + 3\frac{(\lambda+2\mu)}{\lambda} v_3^{2(k+1)} = \Delta \bar{v}_3^{(k+1)} + \frac{\epsilon^2}{2\mu} F_3^0 & \omega, \\ v_3^2 = 0 & \text{on 7.} \end{cases}$$

5.6 . Tau Approximation of Order 3: Numerical Results.

In order to estimate the error introduced by a Tau approximation of order 3 we chose the shape of the plate and the applied forces so that an explicit solution to the two-dimensional problem is known.

The middle surface of the plate is a square of length 1, w $=]0,1[\times]0,1[$. The Young modulus $E = 21\ 10^5$, and Poisson ration $\nu = 0.3$ correspond to the parameters of a steel material.

The horizontal components of the body force and the surface forces are equal to zero, $f_\alpha = 0$, $1 < \alpha < 3$, $g_i = 0$, $1 < i < 3$.

The vertical component of the body force is such that the solution to the two-dimensional problem is $[x(x-1)y(y-1)]^2$.

The following table gives the relative error in H^1 norm between the solutions of the three-dimensional and two-dimensional problems on the middle surface as a function of the thickness of the plate (2ϵ).

thickness (2ϵ)	relative error in H^1 norm %
0.002	0.02
0.004	0.04
0.006	0.08
0.008	0.15
0.010	0.24
0.012	0.35
0.014	0.47
0.016	0.61
0.018	0.77
0.020	0.95
0.040	3.55
0.060	7.26

Computations have been made with a regular mesh size of $\frac{1}{100}$ in both directions of the plate, and have been performed on the ALLIANT CONCENTRIX using the-ARGONNE library routines (FISH-PACK and BIHAR).

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