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Mesh-independent Spectra in the Moving Finite Element Equations

by

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Abstract

We derive the Moving Finite Element (MFE) equations for the solution of a scalar evolutionary equation in d space dimensions ($d \geq 1$) and introduce the elementwise approach to MFE. This approach yields a decomposition of the mesh- and solution-dependent matrix A in the (semi-discretised) non-linear system of ordinary differential equations $A(y)\dot{y} = g(y)$ which forms the basis for proofs of eigenvalue clustering. With a simple, specific block diagonal preconditioner, D, it is shown that the eigenvalue spectum of the preconditioned MFE matrix $D^{-1}A$ is $[\frac{1}{2}, 1 + \frac{d}{2}]$ indcpcndcntly of the mesh configuration, the solution and the number of nodes. A more specific result is established for the case $d = 1$. These results guarentee extremely rapid solution techniques using, for example conjugate gradient methods. We show how the analysis extends to systems of partial differential equations when a separate moving mesh is used for each component.

1.Introduction

The Moving Finite Element (MFE) Method, introduced by Miller and Miller [11], has been used with considerable success to obtain solutions to a number of time-dependent partial differential equations - see [5],[7],[8] and [10] for parabolic problems and [14],[15]-and [13] for hyperbolic problems. In this paper we present no numerical results, but rather describe and extend previous theoretical work ([14]) to prove results on the inherent good conditioning of the MFE equations independently of the solution, the mesh configuration and the number of nodes.

Moving and adaptive mesh methods for the numerical solution of time-dependent partial differential equations are developed because of the possibility of accurate representation of true solutions on grids which distort to describe developing or propergating features. In most such methods the equations for determining grid configuration are considered separately or are decoupled from those for the solution representation (see for example [1],[2],[4] or the review in [G]). The MFE method however goes against this tradition in that simultaneous equations for the solution and mesh variables are derived simply on the basis of minimisation of a residual norm. In this paper we show that t he resulting MFE equations have a definite structure, and t hat the algebraic conditioning of the equations is good this result being independent of the mesh-configuration, the solution and the number of nodes. These results do not hold if the solution and mesh variables are considered separately.

In section 2 we describe the piecewise linear MFE method without penalty terms for a scalar evolutionary equation in d-dimensional space ($d \geq 1$). This yields the MFE equations in the form

$$A(y)\dot{y} = g(y) \tag{1.1}$$

where y is a vector containing the unknowns (the finite element coeficients and node position vectors), A is the solution- and mesh-dependent MFE matrix and g comes from the spatial derivat ive terms. In section 3 we briefly consider appropriate modelling of boundary conditions. In section 4 (for t he one-dimensional case) and section 6 (for higher dimensions) we introduce the elementwise approach to MFE and show how this yields the structure of the MFE equations through a decomposition of the matrix A. This decomposition explicitly reveals the causes of indeterminacy in the MFE method and allows simple and natural treatment of these degeneracies without the need for penalty terms. These procedures are described fully in [14] and

[13] (see also [15]) and are not described again here. The main results of this paper are presented in sections 5 (one-dimension) and 7 (higher dimensions) where wc prove that if wc premultiply the MFE matrix A(y) by the inverse of its $(d+1)$-square diagonal blocks D(y), then the eigenvalues of the resulting matrix $D^{-1}A$ lie in the closed interval $[\frac{1}{2}, 1+\frac{d}{2}]$ independently of the solution, the mesh configuration and the number of nodes. In one-dimension the result is more specific : $D^{-1}A$ has just two distinct eigenvalues of $\frac{1}{2}$ and $\frac{3}{2}$ and possibly one or two of unity depending on boundary conditions. There results contrast with the sitaution in standard (fixed mesh) finite element methods where the conditioning of mass or stiffness matrices are certainly mesh-dependent (see for example [12, p.213]). In section 8 we show how these results extend to systems of partial differential equations if a separate moving mesh is used for each component of the system.

2. The Moving Finite Element Method

In this section we develop the basic piecewise linear Moving Finite Element (MFE) method for a scalar evolutionary equation of the form

$$u_t - L(u) = 0 \tag{2.1}$$

together with appropriate initial and boundary conditions in a region Ω of d dimensional physical space, and some time interval $[0,T]$. We use simplex elements (i.e. simple line segments in 1 dimension, triangles in 2 dimensions, tetrahedra in 3 dimensions,...) with nodes at the vertices and thus seek an approximation of (2.1) of the form

$$U(\underline{r},t) = \sum_{j=1}^{N+B} a_j(t)\alpha_j(\underline{r},\underline{s}(t)) \tag{2.2}$$

where $\underline{r}$ is the position vector of a point,

$$\underline{s} = (\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_{N+B})^T \tag{2.3}$$

contains the nodal position vectors $\underline{s}_j$, a_j is a nodal parameter, and α_j is a finite element basis function. Here N is the number of internal nodes and B the number of boundary nodes. Any of these nodes may have any number of the variables of $\underline{s}_j$ and/or a_j constrained to be held fixed, and modelling of boundary conditions is achieved by applying such constraints appropriately. (We return to the issue of boundary conditions in section 3). In the following we assume (for definiteness) that all boundary nodes are held fixed as well as

the amplitudes at these nodes — this models Dirichlet conditions over a fixed domain. The function $U(\underline{r}, t)$ thus has $N(1 + d)$ degrees of freedom.

Partial differentiation of (2.2) gives

$$U_t(\underline{r},t) = \sum_{j=1}^{N} \dot{a}_j(t)\frac{\partial U}{\partial a_j} + \dot{\underline{s}}_j(t).\nabla_{\underline{s}_j} U$$
$$= \sum_{j=1}^{N}\left[\dot{a}_j\alpha_j(\underline{r},\underline{s}(t)) + \sum_{m=1}^{d}\dot{s}_{jm}(t)\beta_{jm}(\underline{r},\mathbf{a}(t),\underline{s}(t))\right] \tag{2.4}$$

whcro $\dot{s}_{jm}$(t) is the m^{th} component Of $\dot{\underline{s}}_j$ (t) and

$$\beta_{jm} = \frac{\partial U}{\partial s_{jm}} \tag{2.5}$$

is one of d second type basis functions. (˙ denotes time derivat ive). The support, of each β_{jm} is the same as that Of α_j and indeed the-arguements Of Lynch [9] can be used to show that

$$\beta_{jm} = -\frac{\partial U}{\partial x_m}\alpha_j, \tag{2.6}$$

where we have taken $\underline{r}$ Of (2.2) to be

$$\underline{r} = (x_1, x_2, \ldots, x_d). \tag{2.7}$$

Note that each β_{jm} is linear on each element, but is in general discontinuous across the element edges which emanate from node j since the derivatives $\partial U/\partial x_m$ are piecewise constant funct ions. Minimisation o f the square of the L_2 norm of the residual

$$\|U_t - L(U)\|^2_{L_2} \tag{2.8}$$

with respect to the time derivatives of the parameters $\dot{a}_j, \dot{s}_{jm}, m = 1, \ldots, d$, $j = 1, \ldots,$ N gives rise to the N(1 + d) equations

$$\begin{aligned}\langle U_t - L(U), \alpha_j \rangle &= 0\\ \langle U_t - L(U), \beta_{jm}\rangle &= 0\end{aligned}, \tag{2.9}$$

$m = 1, \ldots,$ if, $j = 1, \ldots, N$. Substituting for U_t from (2.4) then gives the set of MFE equations

$$A(y)\dot{y} = g(y) \tag{2.10}$$

where

$$y = (a_1, \underline{s}_1^T; a_2, \underline{s}_2^T; \ldots; a_N, \underline{s}_N^T)^T \tag{2.11}$$

and $g(y)$ arises from the terms containing L(U) in (2.9).

The MFE matrix A is square and symmetric, consisting of inner products of the a's and β's in blocks, the $(i,j)^{th}$ $(d+1)$ x $(d+1)$ block being

$$A_{ij} = \begin{pmatrix} \langle \alpha_i, \alpha_j \rangle & \langle \beta_{i1}, \alpha_j \rangle & \dots & \langle \beta_{id}, \alpha_j \rangle \\ \langle \alpha_i, \beta_{j1} \rangle & \langle \beta_{i1}, \beta_{j1} \rangle & \dots & \langle \beta_{id}, \beta_{j1} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \alpha_i, \beta_{jd} \rangle & \langle \beta_{i1}, \beta_{jd} \rangle & \dots & \langle \beta_{id}, \beta_{jd} \rangle \end{pmatrix}$$

where $\langle ., . \rangle$ denotes the L_2 inner product,. Note that so long as U_t remains square integrable, we have that

$$\|U_t\|^2_{L_2} = \dot{y}^T A \dot{y} \tag{2.13}$$

from which follows that the MFE matrix is positive semi definite and singular only when $\dot{y} \neq 0$ exists such that $U_t = 0$. We will further mention the potential causes of singularity of A in sections 4 and 6, but refer to [14],[15] and [13] for a complete description.

3. Boundary Conditions

There is considerable freedom in the manner in which boundary conditions can be modelled in the MFE method. Essentially we may hold fixed or allow to vary in the minimisation (2.8) any subset of the parameters

$$\{a_i, \underline{s}_i^T : i = N+1, \dots, N+B\}, \tag{3.1}$$

to achieve the appropriate conditions.

A Dirichlet condition

$$u = h(r) \tag{3.2}$$

on the boundary $\partial\Omega$ of the fixed region Ω could be modelled by fixing each $\underline{s}_i$ to remain at it's initial position on $\partial\Omega$ and setting $a_i = h(\underline{s}_i)$ for all time. It may in certain problems however be appropriate to allow some or all of the boundary nodes to move along the boundary. Depending on the geometry of the boundary (and the coordinate system used) this may be easy or more difficult to achieve in practice.

For a homogeneous Neumann condition,

$$\frac{\partial u}{\partial n} = 0 \qquad on\ \partial\Omega, \tag{3.3}$$

the variables $\dot{a}_i$, $i = N+1, \ldots, N+B$ are unconstrained and are determined from the minimization of the residual (2.8). Except only for degencracies that might arise-these arc described precisely in [14],[15] and [13]—the positions of boundary nodes may also be allowed to vary in the minimiantion of (2.8). Thus some part or all of the boundary may bc allowed to move as the solution evolves.

4. Elementwise Approach and Equation Structure-One-Dimension

In one-dimension ($d = 1$) the obvious choisc of node numbering is

$$a = s_0 < s_1 < \cdots < s_N < s_{N+1} = b \tag{4.1}$$

where Ω is the interval $[a, b]$ and s_0 and s_{N+1} arc the boundary nodes. (Note that though this is the natural ordering in one-dimension, the numbering of boundary nodes differs from that used in (2.2) which will bc used in higher dimensions).

Wc introduce some numbering of the elements $k = 1, \ldots, n = N + 1$ (n is the total number of elements and N is the total number of internal nodes as before). For each element k, let $\nu = 1, 2$ number the two nodes (vertices) o f the element. Then we define $\phi_{k\nu}$ to be the 'element basis funct ion which has support only on element k and which is linear, taking the value 1 at node ν and zero at the other node of element k. In this one-dimensional setting there is a simple choice of the element numbering in terms of the node numbering given by (4.1), namely that the element between nodes i and $i + 1$ is numbered $k = i + 1$, $\nu = 1$ denoting the left hand vertex which is node i and $\nu = 2$ the right hand node $i + 1$. For $d = 1$, α_i is the simple hat function with support on just the two elements about the node i, thus with the element numbering described above, ϕ_{k1} is that part of the basis function α_i on element $k = i + 1$, and ϕ_{k2} is that part of α_{i+1} on the same element.

With these definitions, we reparameterise the piecewise linear (discontinuous) function $U_l(x, t)$ of (2.4) in the form

$$U_l(x,t) = \sum_{k=1}^{n} \sum_{\nu=1}^{2} \dot{w}_{k\nu}(t)\phi_{k\nu}(x, s(t)). \tag{4.2}$$

To be consistent with the Dirichlet boundary conditions of section 2 we must impose

$$\dot{w}_{11} = 0 = \dot{w}_{n2} \tag{4.3}$$

and thus write

$$U_t = \sum_{k=2}^{n} \dot{w}_{k1}\phi_{k1} + \sum_{k=1}^{n-1} \dot{w}_{k2}\phi_{k2}. \tag{4.4}$$

From (2.4) (with $d = 1$) and (4.4) we may relate the nodal parameters $\dot{\mathbf{a}}, \dot{\mathbf{s}}$ to the $\dot{w}_{k\nu}$ as follows.From *(2.4)* and (2.6)

$$U_t = \sum_{j=1}^{N} \dot{a}_j\alpha_j + \dot{s}_j\beta_j = \sum_{j=1}^{N} (\dot{a}_j - U_x\dot{s}_j)\alpha_j, \tag{4.5}$$

and with the simple choice of node-element numbering described above we have

$$\alpha_i = \phi_{i2} + \phi_{i+1\,1}; \tag{4.6}$$

thus *(4.5)* is

$$U_t = \sum_{j=1}^{N} (\dot{a}_j - U_x\dot{s}_j)(\phi_{j2} + \phi_{j+1\,1}). \tag{4.7}$$

Rewriting (4.7) as

$$U_t = \sum_{j=2}^{n=N+1} (\dot{a}_{j\ 1} - m_{j\,\frac{1}{2}}\,\dot{s}_{j-1})\phi_{j1} + \sum_{j=1}^{n-1=N} (\dot{a}_j - m_{j\,,\,\frac{1}{2}}\,\dot{s}_j)\phi_{j2}, \tag{4.8}$$

(where we have written the gradient of U as

$$m_{j+\frac{1}{2}} = \frac{a_{j+1} - a_j}{s_{j+1} - s_j}) \tag{4.9}$$

we can compare directly with (4.4) to give

$$\dot{w}_{i1} = \dot{a}_{i-1} - m_{i-\frac{1}{2}}\dot{s}_{i-1} \qquad \textit{and} \qquad \dot{w}_{i2} = \dot{a}_i - m_{i-\frac{1}{2}}\dot{s}_i. \tag{4.10}$$

Thus if we write these equations as

$$\dot{w} = M\dot{y}$$

where

$$\dot{w} = (\dot{w}_{12}; \dot{w}_{21}, \dot{w}_{22}; \ldots; \dot{w}_{n1})^{\mathsf{T}} \tag{4.12}$$

and $\dot{y}$ is the derivative of (2.11), then the $2N \times 2N$ matrix M in seen to be 2×2 block diagonal with i^{th} diagonal block

$$M_i = \begin{pmatrix} 1 & -m_{i-\frac{1}{2}} \\ 1 & -m_{i+\frac{1}{2}} \end{pmatrix} \quad , i = 1, \ldots, N. \tag{4.13}$$

We note that this block diagonal structure of M is a consequence of the simple $1-1$ node-clement numbering which we have used in the one-dimensional case. (Other orderings of the variables would give rise to a permutation of the rows and columns of M). The situation in higher dimensions when such orderings are not possible will be explained in section 6.

We can also express a simple relationship between the element basis functions $\phi_{k\nu}$ and the node basis functions $\alpha_i, \beta_i \, (\equiv \beta_{i1})$ in terms of the matrix M. To see this, write

$$\alpha = (\alpha_1, \beta_1; \alpha_2, \beta_2; \ldots; \alpha_N, \beta_N)^T \tag{4.14}$$

and

$$\phi = (\phi_{12}; \phi_{21}, \phi_{22}; \ldots; \phi_{n1})^T \tag{4.15}$$

and note that *(2.4) and (4.4)* are

$$\dot{y}^T \alpha = U_l = \dot{w}^T \phi. \tag{4.16}$$

Using (4.1) this yields

$$a = M^T \phi. \tag{4.17}$$

We now introduce the elementwise MFE matrix. In a similar manner to (2. 13), but using the elementwise description (4.4) we obtain

$$\|U_l\|^2_{L_2} = \dot{w}^T C \dot{w} \tag{4.18}$$

where the matrix C has entries which are the inner products of elementwise basis functions $\phi_{k\nu}$ in pairs . Because the support O f each $\phi_{k\nu}$ is on one element only and therare only two different basis functions on each element, C is a block diagonal matrix having at the upper left and lower right corners respectively the 1 x 1 blocks

$$C_1 = \langle \phi_{12}, \phi_{12} \rangle \text{ and } C_n = \langle \phi_{n1}, \phi_{n1} \rangle, \tag{4.10}$$

and otherwise having $n - 2$ (= N − 1) 2 x 2 diagonal blocks, the $(k-1)^{th}$ of which is

$$C_k = \begin{pmatrix} \langle \phi_{k1}, \phi_{k1} \rangle & \langle \phi_{k1}, \phi_{k2} \rangle \\ \langle \phi_{k2}, \phi_{k1} \rangle & \langle \phi_{k2}, \phi_{k2} \rangle \end{pmatrix}. \tag{4.20}$$

Evaluation of these inner products yields

$$C_1 = \frac{1}{3}\Delta s_1 ,$$
$$C_k = \frac{1}{6}\Delta s_k \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} , \quad k = 2 , \ldots , \text{n-1}, \tag{4.21}$$
$$C_n = \frac{1}{3}\Delta s_{N+1} .$$

Note that M and C are of the same dimension, but that the diagonal blocks of C are staggered with respect to those of M.

Using (2.13),(4.11) and (4.18) we have

$$\dot{y}^T A \dot{y} = \|U_t\|^2_{L_2} = \dot{w}^T C \dot{w} = \dot{y}^T M^T C M \dot{y} \tag{4.22}$$

which gives the important equality

$$A = M^T C M , \tag{4.23}$$

where A is the 2 x 2 block triple-diagonal MFE matrix given by (2.12) for cl = 1, M is the 2 x 2 block diagonal matrix given by (4.13)which represents the mapping from the node-based to the element based representations, and C is t he elementwise MFE matrix, also block diagonal and given by (4.21).

A s described in section 3, in the case of homogeneous Neumann boundary conditions we have the nodewise description

$$U_t = \dot{a}_0 \alpha_0 + \sum_{j=1}^{N} \left[\dot{a}_j \alpha_j + \dot{s}_j \beta_j \right] + \dot{a}_{N+1} \alpha_{N+1} \tag{4.24}$$

which has corresponding elementwise description

$$U_t = \sum_{k=1}^{n} \dot{w}_{k1} \phi_{k1} + \sum_{k=1}^{n} \dot{w}_{k2} \phi_{k2} . \tag{4.25}$$

The analysis as above in the Dirichlet case then yields a matrix M which is as in (4.13), but with extra 1 x 1 upper and lower diagonal corner blocks

$$M_0 = 1 \text{ and } M_{N+1} = 1 , \tag{4.26}$$

and a matrix C which is 2 x 2 block diagonal throughout with

$$C_k = \frac{1}{6}\Delta s_k \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad , k = 1, \ldots , \text{n} = N + 1. \tag{4.27}$$

With these modifications, the decomposition (4.23) of the MFE matrix in the Neumann case remains true.

Having established the decomposition (4.23) for A, we are immccliatcly able to deduce that A is non-singular if and only if both M and C are non-singular. From (4.12) and (4.20) the sources of dcgcncracy are thus seen to bc respectively lack of curvature at a nodc (this has been called 'parallelism' in [14] ,[15] and [13]) and clcmcnt-folding ('node-colliding' in one-dimension).

It is not the purpose of this paper to discuss the sources of degeneracy and methods of treatment since this has been covered in [14]. Rather the results of this paper arc proved only for the case of non-singular MFE matrix A.

5. Eigenvalue Clustering--0nc Dimension

Evaluating the inner products in (2.12) we obtain the non-zero blocks of A in the one-dimensional case as

$$A_{i,i} = \frac{1}{3}\begin{pmatrix} \Delta s_i + \Delta s_{i+1} & -\Delta a_i - \Delta a_{i+1} \\ -\Delta a_i - \Delta a_{i+1} & m_{i-\frac{1}{2}}\Delta a_i + m_{i+\frac{1}{2}}\Delta a_{i+1} \end{pmatrix}, \; i = 1,\ldots, N, \tag{5.1}$$

$$A_{i,i-1} = \frac{1}{6}\begin{pmatrix} \Delta s_i & -\Delta a_i \\ -\Delta a_i & m_{i-\frac{1}{2}}\Delta a_i \end{pmatrix}, \; i = 2,\ldots,N \tag{5.2}$$

and

$$A_{i-1,i} = A_{i,i-1} \qquad , i = 2,\ldots,N \tag{5.3}$$

where

$$\Delta a_j = a_j - a_{j-1}.$$

Note that

$$A_{i,i} = 2(A_{i,i-1} + A_{i,i+1}). \tag{5.4}$$

We have

$$det(A_{i,i}) = \frac{1}{9}\Delta s_i \Delta s_{i+1}(m_{i+\frac{1}{2}} - m_{i-\frac{1}{2}})^2 \geq 0 \tag{5.5}$$

with equality if and only if there is parallelism and/or element folding. We assume that neither of these degeneracies occur, so that from the decomposition (4.23), M is non-singular and C is positive definite. Let D be t he positive definite matrix of the 2 x 2 diagonal blocks of A. For convenience we introduce also the

(line) diagonal matrix R where

$$R^2 = \frac{1}{6}\begin{pmatrix} \Delta s_1 & & & 0 \\ & \Delta s_2 & & \\ & & \Delta s_2 & \\ 0 & & & \Delta s_{N+1} \end{pmatrix} \tag{5.6}$$

which is such that

$$C = R\overline{C}R \tag{5.7}$$

with

$$\overline{C} = \begin{pmatrix} 2 & & & & & & & \\ & 2 & 1 & & & & 0 & \\ & 1 & 2 & & & & & \\ & & & 2 & 1 & & & \\ & & & 1 & 2 & & & \\ & & & & & \ddots & & \\ & 0 & & & & & 2 & 1 \\ & & & & & & 1 & 2 \\ & & & & & & & & 2 \end{pmatrix}. \tag{5.8}$$

We then write (4.23) as

$$A = \widetilde{M}^T \overline{C} \widetilde{M} \tag{5.9}$$

where

$$\widetilde{M} = RM . \tag{5.10}$$

Note that $\widetilde{M}$ and $\overline{C}$ have the same structure as M and C respectively since R merely scales the non-zero elements, and that $\widetilde{M}$ is non-singular. Now treating all these matricies as consisting of 2×2 blocks throughout (thus $\overline{C}$ has 2×2 diagonal blocks $\overline{C}_{i,i} = 2I$ and off-diagonal blocks containing the entries of unity), the $(i,j)^{th}$ block of A is

$$A_{i,j} = \widetilde{M}_i^T \overline{C}_{i,j} \widetilde{M}_j \tag{5.11}$$

since $\widetilde{M}$ is block diagonal in this context, $\widetilde{M}_j$ denoting its j^{th} diagonal block. From (5.11) we have

$$A_{i,i} = \widetilde{M}_i^T \overline{C}_{i,i} \widetilde{M}_i , \tag{5.12}$$

so that

$$(A - \lambda D)_{i,j} = \widetilde{M}_i^T (I - \lambda\delta_{ij})\overline{C}_{i,j} \widetilde{M}_j \tag{5.13}$$

where δ_{ij} is the Kronecker delta. We write (5.13) as

$$A - \lambda D = \widetilde{M}^T C_\lambda \widetilde{M} \tag{5.14}$$

where C_λ is the matrix with blocks

$$(C_\lambda)_{i,j} = (1 - \lambda\delta_{ij})\,\overline{C}_{i,j}. \tag{5.15}$$

From (5.14) we have the result that μ is an eigenvalue of $D^{-1}A$ if and only if C_μ is singular. Now

$$C_\lambda = \begin{pmatrix} 2(1-\lambda) & 1 & & & & & \\ 1 & 2(1-\lambda) & 1 & & & & \\ & 1 & 2(1-\lambda) & 1 & & & \\ & & 1 & 2(1-\lambda) & \ddots & & \\ & & & \ddots & 2(1-A) & 1 & \\ & & & & 1 & 2(1-\lambda) & 2(1-\lambda) \end{pmatrix}; \tag{5.16}$$

thus since

$$\begin{vmatrix} 2(1-\lambda) & 2(1-A) \\ 1 & 2(1-\lambda) \end{vmatrix} = (2\lambda - 1)(2\lambda - 3) \tag{5.17}$$

we have that the eigenvalues of $D^{-1}A$ are $N - 1$ of $\frac{1}{2}$, 2 of unity and $N - 1$ of $\frac{3}{2}$. The eigenvalues of $D^{-1}A$ are data independent (though those of A and D separately are certainly not).

For homogeneous Neumann conditions a similar argument to the above yields a matrix C_λ which is block diagonal throughout, all diagonal blocks being

$$\begin{pmatrix} 2(1-\lambda) & 1 \\ 1 & 2(1-A) \end{pmatrix}. \tag{5.18}$$

Thus for these boundary conditions the eigenvalues of $D^{-1}A$ are all $\frac{1}{2}$ and $\frac{3}{2}$ in pairs.

We emphasise the mesh- and solution-independence of these results which may be used to give extremely rapid inversion techniques. In particular the author has used (in [14],[15] and [13]) the 'Generalised Conjugate Gradient Method' of Concus, Golub and O'Leary [3] for which the exact solution of a MFE linear system is guarenteed after just 2 iterations if there are only two distinct eigenvalues for $D^{-1}A$ (the Neumann case), or 3 iterations if there are Dirichlet boundary conditions which (as shown above) give rise to just one extra distinct eigenvalue.

6. Elementwise Approach and Equation Structure—Higher Dimensions

The development we give here parallels that given for the one-dimensional case in section 4 but contains some important differences.

We choose some numbering of the elements $k = 1, \ldots, n$ and some numbering $\nu = 1, \ldots, d+1$ of the vertices of each element, and introduce linear element basis functions $\phi_{k\nu}$ which have support only on

element k , taking the value 1 at node ν of that element, and zero at all other nodes. We then reparameterise the piecewise linear (discontinuous) function U_t of (2.4) as

$$U_t(\underline{r}, t) = \sum_{k=1}^{n} \sum_{\nu=1}^{d+1} \dot{w}_{k\nu}(t)\phi_{k\nu}(\underline{r}, \underline{s}(t)) \tag{6.1}$$

where, to be consistent with the assumed Dirichlet conditions in (2.4) we take

$$\dot{w}_{k\nu} = 0 \tag{6.2}$$

whenever the ν^{th} node of element k is one of the boundary nodes with position $\underline{s}_j$, $j = N + 1, \ldots, N + B$: equivalently we consider the $k\nu^{th}$ term of (6.1) to be removed. Thus for $d \geq 2$ we shall assume that (6.1) implicitly includes the effect of boundary conditions since (unlike in the one-dimensional case) no simple node-element numbering exists which facilitates the identification of boundary nodes. From (6.1) we have

$$\|U_t\|_{L_2}^2 = \dot{w}^T C \dot{w} \tag{6.3}$$

where

$$\dot{w} = (\dot{w}_{1\,1}, \ldots, \dot{w}_{1\,d+1}; \ldots; \dot{w}_{n\,1}, \ldots, \dot{w}_{n\,d+1})^T \tag{6.4}$$

(boundary terms having been removed) and C is the square elementwise matrix having entries which are the inner products

$$\langle \phi_{k\nu}, \phi_{l\mu} \rangle \tag{6.5}$$

of elementwise basis functions in pairs. If we order the basis functions as in (6.4) ,namely

$$\phi = (\phi_{1\,1}, \ldots, \phi_{1\,d+1}; \ldots; \phi_{n\,1}, \ldots, \phi_{n\,d+1})^T, \tag{6.6}$$

then because of the single element support of each $\phi_{k\nu}$, C is a block diagonal matrix. Each block is $(d + 1)$ x (d + 1) (or because of boundary conditions $(d + 1 - q)$ x $(d + 1 - q)$ if an element has q vertices on the boundary) , the entries in the k^{th} block being a simple scalar multiple of the measure of element k. Furthermore,for the piecewise linear elements on simplicies that we are considering, it can be readily shown that for all $\mu \neq \nu$

$$\langle \phi_{k\nu}, \phi_{k\nu} \rangle = 2\langle \phi_{k\nu}, \phi_{k\mu} \rangle . \tag{6.7}$$

Thus the k^{th} block of C is

$$C_k = \sigma_d R_k (I + E) \tag{6.8}$$

where E is the matrix with all entries equal to 1, I is the itlcntity matrix, R_k is the measure of element k, and σ_d is a positive constant dependent only on the dimension of the physical space (c.f. (4.21) or (4.27) in the one-dimensional case). Note that the effect of boundary constraints is simply to reduce the dimension of C_k and not to alter the structure (6.8).

We may also relate the nodal parameters $\dot{y}$ to the elementwise parameters $\dot{w}$ by

$$\dot{w} = M \dot{y} \tag{6.9}$$

where M is the rectangular matrix obtained by writing the α_j, $\underline{\beta}_j [\equiv (\beta_{j\,1}, \ldots, \beta_{j\,d})]$ in terms of the $\phi_{k\nu}$. To see this we have

$$U_t = \sum_{j=1}^{N} (\dot{a}_j \alpha_j + \underline{\beta}_j \cdot \underline{\dot{s}}_j) = \dot{y}^T \alpha \tag{6.10}$$

where

$$\alpha = (\alpha_1, \underline{\beta}_1^T; \ldots; \alpha_N, \underline{\beta}_N^T)^T \tag{6.11}$$

and using (2.6),

$$U_t = \sum_{j=1}^{N} (\dot{a}_j - \nabla U . \underline{\dot{s}}_j) \alpha_j = \dot{w}^T \phi \tag{6.12}$$

since each α_j is the sum of some set of $\phi_{k\nu}$. It follows from (6.9),(6.10) and (6.12) that

$$\mathrm{a} = M^T \phi \tag{6.13}$$

as in the one-dimensional case. Since α_j and $\underline{\beta}_j$ are defined on the support of a local patch of elements around the node j, M has a relatively simple structure, its entries involving only the components of the (constant) gradient of the solution in the various elements.

Equations (2.13),(6.3) and (6.9) together imply that

$$A = M^T C M . \tag{6.14}$$

Since $\dot{y}^T A \dot{y} = 0$ if and only if $\dot{w}^T C \dot{w} = 0$ with $\dot{w} = M \dot{y}$, it follows that A is singular only if C is singular or the rectangular matrix M is column rank deficient. Singularity of C occurs only when the measure of

an element decreases to zero -i.e. only when there is element-folding. Column rank deficiency of M is the consequence of lack of curvature i.e. parallelism. As in the one-dimensional case these degeneracies are described completely in [14] and [13] and it is not the purpose of this paper to discuss these further.

To find the form of $\mathbf{M}$ from (6.9) consider a single element l. At the vertex μ of this element which is node i say, we compare U_l in the nodewise and elementwise bases represented in the equations

$$U_l = \sum_{j=1}^{N}\left[\dot{a}_j\alpha_j + \sum_{m=1}^{d}\dot{s}_{jm}\beta_{jm}\right] = \sum_{k=1}^{n}\sum_{\nu=1}^{d}\dot{w}_{k\nu}\phi_{k\nu} \tag{6.15}$$

to obtain

$$\dot{w}_{l\mu}\phi_{l\mu} = \dot{a}_i\alpha_i + \sum_{m=1}^{d}\dot{s}_{im}\beta_{im} \tag{6.11}$$

$$= \left[\dot{a}_i - \sum_{m=1}^{d}\frac{\partial U}{\partial x_m}\dot{s}_{im}\right]\alpha_i \tag{6.17}$$

using (2.G). (Note that for a boundary node i both sides of (6.16) will be zero for the assumed Dirichlet conditions). Thus since on element l

$$\phi_{l\mu} = \alpha_i \quad ,$$

we have

$$\dot{w}_{l\mu} = \dot{a}_i - \sum_{m=1}^{d}\frac{\partial U}{\partial x_m}\dot{s}_{im} \tag{6.18}$$

so that in the row of M corresponding to $\dot{w}_{l\mu}$ there is just one $(d+1)$ row-vector entry

$$\left(1, -\frac{\partial U}{\partial x_1}, \ldots, -\frac{\partial U}{\partial x_d}\right) = \underline{p}_l^T \tag{6.19}$$

say, in the i^{th} blocked column : note that $\dot{y}$ is given by (2.11) and each $\underline{s}_i$ is assumed ordered as

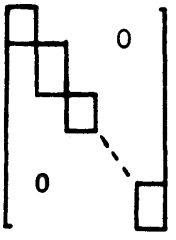
$$\underline{s}_i = (s_{i1}, s_{i2}, \ldots, s_{id})^T \quad , \tag{6.20}$$

s_{ij} corresponding to the j^{th} coordinate x_j.

There will be a similar $(d+1)$-vector entry in the i^{th} column for each element k surrounding node i : thus there will be the same number of vectors of the type $\underline{p}_k^T$ in column i as there are elements around node i. Indeed if we order the vector $\dot{w}$ not as in the elementwise manner of (6.4) (which is illustrated for the two-dimensional case in figure (6.1)), but in a nodewise manner (as illustrated for the two-dimensional case

in figure (6.2)), then the rows of the matrix M will be interchanged in such a way that the resulting matrix is block diagonal. That is, there exists a permutation matrix Q such that QM = N, say, is block diagonal with rectangular blocks, i.c.

$$Q\,M = N = \qquad (6.21)$$



where each rectangular block consists of $(d + 1)$ columns of scalars (corresponding to the components of $\underline{p}_k^T$) and the same number of rows as there are elements surrounding a node.

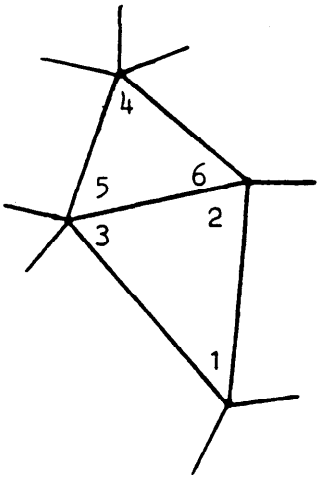


Figure (G.l) : Elementwise Numbering

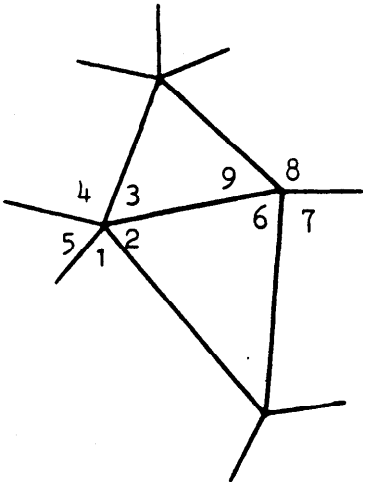


Figure (6.2) : Nodcwise Numbering

For other boundary conditions mentioned in section 3, the decomposition (G. 14) remains true, though the matrix C (as described previously) and the matrix M are slightly different than that described above. For a homogeneous Neumann condition on a fixed domain with fixed boundary nodes ($s_{im} = 0$ for $m = 1, \ldots, d$, i = N + I,... , N + B), the equation (G.18) becomes

$$\dot{w}_{l\mu} = \dot{a}_i \qquad (G.22)$$

so that M has row entries

$$1 = p_l^T \qquad (6.23)$$

in the i^{th} blocked column. Similarly for other boundary constraints, the vector $\underline{p}_i^T$ of (6.19) will be of length less than $(d + 1)$ when i is a boundary node. That is, boundary constraints merely reduce the number of entries in $\underline{p}_i^T$, but the structure as described above otherwise remains true.

7. Eigenvalue Clustering - Higher Dimensions

The solution-independent eigenvalue clustering results given in section 5 for the MFE method in one-dimension extend to higher dimensional problems. We give a proof in this section of the general result.

As in section 5 we assume that there is no parallelism and no clustering-folding.

We give the proof first for the case in which there are no constraints on the amplitudes nor position vectors of any boundary nodes. This assumption merely simplifies the proof. At the end of the section we shall describe how the proof is simply ammended to establish the result for all types of boundary conditions mentioned in section 3.

We let D be the positive definite matrix of diagonal blocks of the MFE matrix A given by (2.12) for $i = j$. To see the definitness of these blocks, let

$$z_i = \dot{a}_i \alpha_i + \sum_{m=1}^{d} \dot{s}_{im} \beta_{im} \tag{7.1}$$

be the (single-valued) function which has only local support around node i. Writing

$$\dot{z} = (\dot{a}_i, \dot{s}_{i1}, \dot{s}_{i2}, \ldots, \dot{s}_{id})^T \tag{7.2}$$

then

$$\|z_i\|_{L_2}^2 = \dot{z}^T A_{ii} \dot{z} \geq 0 \tag{7.3}$$

with equality if and only if $\dot{z} \neq 0$ exists such that $z_i = 0$. There is no parallelism, the inequality in (7.3) is therefore strict and we have the required definiteness.

The result we shall prove is that independently of the mesh configuration and the number of nodes the positive definite matrix $D^{-1}A$ has no eigenvalues greater than $1 + \frac{d}{2}$ and none less than $\frac{1}{2}$.

As the first step in the proof we obtain a decomposition of the matrix $A - \lambda D$ as follows. Introduce the (line) diagonal matrix R defined by

$$R^2 = \sigma_d \begin{pmatrix} R_1 & & & & & & \\ & \ddots & & & & & \\ & & R_1 & & & & \\ & & & R_2 & & & \\ & & & & \ddots & & \\ & & & & & R_n & \\ & & & & & & \ddots \\ & & & & & & & R_n \end{pmatrix} \tag{7.4}$$

where each diagonal block is $(d + 1)$ x $(d + 1)$, R_k (> 0) is the measure of element k $(k = 1, \ldots, n)$, and σ_d is the positive constant given in (G.8). Then the elementwise MFE matrix C satisfies

$$C = R \bar{C} R \tag{7.5}$$

where (SW again (6.8))

$$\bar{C} = \begin{pmatrix} 2 & 1 & \cdots & 1 & & & & & & & & \\ 1 & 2 & & & & & & & & & & \\ \vdots & & \ddots & \vdots & & & & & & & & \\ 1 & \cdots & 1 & 2 & & & & & & & & \\ & & & & 2 & 1 & \cdots & 1 & & & & \\ & & & & 1 & 2 & & & & & & \\ & & & & \vdots & & \ddots & \vdots & & & & \\ & & & & 1 & \cdots & 1 & 2 & & & & \\ & & & & & & & & \ddots & & & \\ & & & & & & & & & 2 & 1 & \cdots & 1 \\ & & & & & & & & & 1 & 2 & & \\ & & & & & & & & & \vdots & & \ddots & \vdots \\ & & & & & & & & & 1 & \cdots & 1 & 2 \end{pmatrix}, \tag{7.6}$$

each block being also $(d + 1)$ square.

Define $\widetilde{M}$ by

$$\widetilde{M} = RM \quad , \tag{7.7}$$

which is a scaling of the rows of the full column rank rectangular matrix M. Note $\widetilde{M}$ is also of full column rank. Let Q be the permutation matrix given in (G.21) which is such that

$$N = QM \tag{7.8}$$

is $(d+1) \times (d+1)$ block diagonal and is negative definite for any given μ if and only if $\tilde{C}_\mu$ is negative definite for the same μ.

Because of the definition of $\bar{C}$ (7.6) and the constructions of $\tilde{C}$, $\tilde{C}_\mu$, and $\bar{C}_\mu$ in (7.12),(7.17) and (7.23) respectively, we have that the blocks of $\bar{C}_\mu$ have all diagonal entries equal to

$$2(1-\mu) , \tag{7.24}$$

and all off-diagonal cntrics of either 1 or $(1-\mu)$. Wc shall now show that without loss of generality the off-diagonal entries may be taken to be 1.

We first construct a particular elementwise numbering for which the off-diagonal entries in the blocks of $\bar{C}_\mu$ are all 1. For a given mesh of simplices select any element and number it's vertices as $1, \ldots, d+1$. Next take any neighbouring element (with d vertices at d nodes in common with the first element) and continue the numbering in such a way that (a)the first vertex numbered $(d+2)$ does not coincide with a vertex in the first element, and (b)the remaining vertices are numbered $(d+2+i)$ where the order $i = 1, \ldots, d$ is determined by the increasing order of the vertices of the previous numbered element. If a neighbouring element to this last numbered element exists then steps (a) and (b) can be repeated in the numbering of this new element, and the process continued. If at any stage no such neighbouring element exists then any new element can be chosen and it's vertices numbered in any manner except only that vertices which coincide with vertices of the immediately previously numbered element must be numbered last in this new element, and in the same order. The steps in this numbering can always be done since each new element must possess at least one vertex distinct from those of the previously numbered element.

The purpose of this construct ion is that any set of $d+1$ consecutive elementwise numbers must refer to vertices at exactly $d+1$ distinct nodes. An example of this type of elementwise numbering is illustrated for the two-dimensional case in Figure (7.1) : compare this with the elementwise ordering illustrated in Figure (G.l) which is not, of the described type since the vertices 3 and 5 (of the consecutive triplet (3,4,5)) are at the same node.

Thus

$$(A - \lambda D)_{ij} = \tilde{N}_i^T (I - \lambda \delta_{ij}) \tilde{C}_{ij} \tilde{N}_j \tag{7.15}$$

where δ_{ij} is the Kronecker δ . We write this as

$$A - \lambda D = \tilde{N}^T \tilde{C}_\lambda \tilde{N} \tag{7.16}$$

whcrc $\tilde{C}_\lambda$ is the matrix blocked in the same manner as $\tilde{C}$ (which is described above) and having (i, j)th block

$$(\tilde{C}_\lambda)_{ij} = (\quad - \quad \lambda \delta_{ij}) \tilde{C}_{ij} \ . \tag{7.17}$$

Having derived the decomposition (7.16) WC now prove the stated result by contradiction. WC firstly establish the upper bound of $1 + \frac{d}{2}$ on the eigenvalues of $D^{-1}A$.

Assume I hat there mists an eigenvalue μ of $D^{-1}A$ which is such that

$$\mu > 1 + \frac{d}{2} \quad . \tag{7.18}$$

Then there exists x $\neq$ 0 such that

$$\mathbf{x}^T (A - \mu D)\mathbf{x} = 0 \quad , \tag{7.19}$$

Using (7. 16) I his is

$$\mathbf{x}^T \tilde{N}^T \tilde{C}_\mu \text{fix} = 0 \tag{7.20}$$

or

$$\mathbf{z}^T \tilde{C}_\mu \mathbf{z} = 0 \tag{7.21}$$

whcrc

$$\mathbf{z} = \tilde{N}\mathbf{x} \tag{7.22}$$

i s non-zero because N has full column rank. We now prove that $\tilde{C}_\mu$ is st rictly negative definite for all $\mu > 1 + \frac{d}{2}$ thus contradicting (7.21).

From (7.17) $\tilde{C}_\mu$ has the same structure as C , and because of the definition (7.12) we have that the matrix $\bar{C}_\mu$ given by

$$\bar{C}_\mu = Q^T \tilde{C}_\mu Q \tag{7.23}$$

where N is block diagonal with rectangular blocks each consisting of $(d + 1)$ columns of scalars and the same number of rows as there are elements around a node. Note that

$$\tilde{N} = Q\tilde{M} \tag{7.9}$$

has the same block diagonal structnrc as N and is also of full column rank. Now from (6.14)

$$\begin{aligned} A &= M^T C M \qquad (7.10)\\ &= M^T R \bar{C} R M \\ &= (RM)^T Q^T Q \bar{C} Q^T Q (RM) \\ &= \tilde{M}^T Q^T Q \bar{C} Q^T Q \tilde{M} \\ &= \tilde{N}^T Q \bar{C} Q^T \tilde{N} \\ &= \tilde{N}^T \tilde{C} \tilde{N} \qquad (7.11) \end{aligned}$$

where

$$\tilde{C} = Q \bar{C} Q^T \tag{7.12}$$

is an orthogonal transformation Of the $n(d + 1)$ x $n(d + 1)$ matrix $\bar{C}$ given by (7.6).

Let the i^{th} diagonal (rectangular) block of $\tilde{N}$ be $\tilde{N}_i$, and block the matrix $\tilde{C}$ in a similar manner , i.e. such that the i^{th} diagonal block $\tilde{C}_{ii}$ is q x q if there are q elements around node i , that is if there are q rows in $\tilde{N}_i$. Then if A_{kl} is the $(k, 1)^{\omega}$ $(d + 1)$ x $(d + 1)$ block of the MFE matrix A , WC have

$$A_{ij} = \tilde{N}_i^T \tilde{C}_{ij} \tilde{N}_j \tag{7.13}$$

because of the diagonal structure of $\tilde{N}$. In particular the diagonal blocks D of A are given by

$$A_{ii} = \tilde{N}_i^T \tilde{C}_{ii} \tilde{N}_i \quad . \tag{7.14}$$

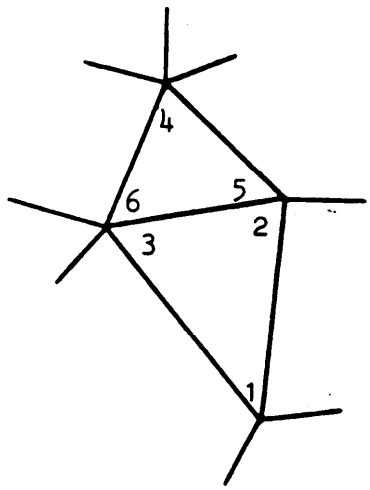


Figure (7.1)

Although this elementwise ordering is used in the proof, since the eigenvalues are independent of the elementwise ordering used, the result is true for any elementwise ordering.

With the above type of elementwise ordering, all off-diagonal terms of the diagonal blocks of $\bar{C}_\mu$ are 1 , since an entry of $(1-\mu)$ only occurs if $(d+1)$ consecutive elementwise numbers represent vertices at less than $(d+1)$ distinct nodes - it is precisely to exclude such an occurence that the ordering above was constructed.

To establish the upper bound of $1+\frac{d}{2}$ it remains only to show that the $(d-1)\times(d+1)$ block

$$(\bar{C}_\mu)_{ii} = \begin{pmatrix} 2(1-\mu) & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 2(1-\mu) \end{pmatrix} \tag{7.25}$$

is negative definite for all $\mu > 1+\frac{d}{2}$. This follows simply by applying the Gershgorin Circle Theorem.

To establish the lower bound we similarly assume that $\mu < \frac{1}{2}$ exists such that $(A-\mu D)$ is singular. This again implies that $z \neq 0$ exists such that (7.21) is true, and we contradict this by showing that $\tilde{C}_\mu$ is strictly positive definite for all $\mu < \frac{1}{2}$. As above this reduces to showing that $(\bar{C}_\mu)_{ii}$ given by (7.25) is strictly positive definite for all $\mu < \frac{1}{2}$. To show this note that for any

$$\mathbf{q} = (q_1, \ldots, q_{d+1})^T \neq 0 \tag{7.26}$$

we have

$$\mathbf{q}^T(\bar{C}_\mu)_{ii}\mathbf{q} = (1-2\mu)\left(\sum_{i=1}^{d+1} q_i^2\right) + \left(\sum_{i=1}^{d+1} q_i\right)^2$$
$$> 0 \quad for \quad \mu < \frac{1}{2} \quad , \tag{7.27}$$

thus giving the rcquircd result.

When boundary constraints are applied, the following considerations allow extension of the proof. Some of the blocks of C (and thus of $\bar{C}$ in (7.6)) may be of dimension less than $(d+1)$, but the structure given in equation (6.8) holds whntcvcr the size of block. The hounds on μ for definiteness of $(C_\mu)_{ii}$ in (7.25) are therefore not violated. We need also consider a particular elementwise numbering of tho desired type (as described above) for the case when there are elements with constrained vertices (boundary nodes). Such vertices are not to be numbered in this proceedure. The required type of numbering can bc constructed in the same way as that described above when there is no 'neighbouring' element.

We therefore have proved that:

The eigenvalue spectrum of $D^{-1}A$ is contained in the closed interval

$$\left[\frac{1}{2}, 1+\frac{d}{2}\right] \tag{7.28}$$

independently of the number of nodes, the mesh configuration and the solution.

In numerical computations in the two-dimensional case we have found these bounds to be sharp.

The results of this section guarentee rapid convergence of techniques such as the Generalised Conjugate Method of Concus,Golub and O'Leary [3].

8.Systems of Equations

For a system Of M evolutionary equations

$$u_t^\ell - L^\ell(u^1, \ldots, u^M) = 0 \qquad , \ell = 1, \ldots, M \tag{8.1}$$

wc show that if the approximations U^ℓ to the components $\ell = 1, \ldots, M$ each have a separate computational grid, then the structure of the MFE equations is the same as in the scalar casand the results given in this paper for the scalar case carry over to systems. This follows since the left hand side term of (2.10)

arises only from the time-derivative in the differential equation, and the time derivatives appear linearly and independently in the system of equations (8.1) also.

With this approach we seek a solution of (8.1) of the form

$$U^\ell(\underline{r},t) = \sum_{j=1}^{N_\ell+B_\ell} a_j^\ell(t)\alpha_j^\ell(\underline{r},\underline{s}^\ell(t)) \qquad ,\ell = 1,\ldots,M \tag{8.2}$$

where a_j^ℓ is the nodal amplitude at the node with position vector $\underline{s}_j^\ell$ for each component $\ell = 1,\ldots,M$. Respectivcly N_ℓ and B_ℓ arc tho numbers of internal and boundary nodes for the component U^ℓ. Differentiating we obtain

$$U_t^\ell = \sum_{j=1}^{N_\ell}\left[\dot{a}_j^\ell(t)\alpha_j^\ell(\underline{r},\underline{s}(t)) + \sum_{m=1}^{d}\dot{s}_{jm}^\ell(t)\beta_{jm}^\ell(\underline{r},\mathbf{a}^\ell(t),\underline{s}^\ell(t))\right] \qquad ,\ell = 1,\ldots,M \tag{8.3}$$

where we have used (2.5) ,(2.6), namely

$$\beta_{jm}^\ell = \frac{\partial U^\ell}{\partial s_{jm}^\ell} = -\frac{\partial U^\ell}{\partial x_m}\alpha_j^\ell \quad . \tag{8.4}$$

Minimisation of

$$\sum_{\ell=1}^{M}\|U_t^\ell - L^\ell(U^1,\ldots,U^M)\|_{L_2}^2 \tag{8.5}$$

with respect to $\dot{a}_j^\ell$, $\dot{s}_{jm}^\ell$, $m = 1,\ldots,d$, $j = 1,\ldots N_\ell$, $\ell = 1,\ldots,M$ then yields the set of MFE equations

$$\sum_{j=1}^{N_\ell}\dot{a}_j^\ell\langle\alpha_j^\ell,\alpha_i^\ell\rangle + \sum_{j=1}^{N_\ell}\sum_{m=1}^{d}\dot{s}_{jm}^\ell\langle\beta_{jm}^\ell,\alpha_i^\ell\rangle = \langle L^\ell(U^1,\ldots,U^M),\alpha_i^\ell\rangle \tag{8.6}$$

for $i = 1,\ldots,N_\ell$, $\ell = 1,\ldots,M$ and

$$\sum_{j=1}^{N_\ell}\dot{a}_j^\ell\langle\alpha_j^\ell,\beta_{ip}^\ell\rangle + \sum_{j=1}^{N_\ell}\sum_{m=1}^{d}\dot{s}_{jm}^\ell\langle\beta_{jm}^\ell,\beta_{ip}^\ell\rangle = \langle L^\ell(U^1,\ldots,U^M),\beta_{ip}^\ell\rangle \tag{8.7}$$

for $i = 1,\ldots,N_\ell$, $p = 1,\ldots,d$ and $\ell = 1,\ldots,M$.

If we now write

$$y^\ell = (a_1^\ell,\underline{s}_1^\ell;\ldots;a_{N_\ell}^\ell,\underline{s}_{N_\ell}^\ell)^T \tag{8.8}$$

then the equations (8.6),(8.7) can be written as M ordinary differential equation systems linked only by their right hand side (non-derivative) terms, namely

$$A(y^\ell)\dot{y}^\ell = g^\ell(y^1,\ldots,y^M) \tag{8.9}$$

for $\ell = 1, \ldots, M$. The structure of the $N_\ell(d+1) \times N_\ell(d+1)$ matrix A of (8.0) is precisely the same as for the scalar case with elements calculated using the nodal amplitudes and positions of the ℓ^{th} component only. The $N_\ell(d+1)$-vector g^ℓ has elements given by

$$g^\ell_{q_i} = \langle L^\ell(U^1, \ldots, U^M), \alpha^\ell_i \rangle \tag{8.10}$$

$$g^\ell_{q_i+m} = \langle L^\ell(U^1, \ldots, U^M), \beta^\ell_{im} \rangle \tag{8.11}$$

where $q_i = (d+1)i - d$.

The integrations in (8.10),(8.11) which involve evaluation of components other than ℓ over elements of the ℓ^{th} component grid can be carried out without too much difficulty using quadrature. This involves obtaining a number of distinct point values of the various components which can be done without the need for interpolation because of the continuous nature of the finite element solution.

An important feature of this approach is that the results presented in this paper for the scalar case also apply to systems.

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