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Abstract

We present a collection of stability results for finite difference approximations to the advection-diffusion equation $u_t = a\, u_x + b\, u_{xx}$. The results are for centered difference schemes in space and include explicit and implicit schemes in time up to fourth order and schemes that use different space and time discretizations for the advective and diffusive terms. The results are derived from a uniform framework based on the Schur-Cohn theory of Simple von Neumann Polynomials and are necessary and sufficient for the stability of the Cauchy problem. Some of the results are believed to be new.

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1. Introduction

The linear advection-diffusion equation:

$$u_t = au_x + bu_{xx}, \qquad b \geq 0, \tag{1}$$

is often used as a model equation in computational physics, partly because it models two of the most basic processes in a physical system, namely advection and diffusion. In this paper, we are interested in the stability analysis of approximation schemes for solving this model equation. An understanding of the stability properties of a computational scheme is important for both theoretical questions of convergence and for practical questions of sensitivity to round-off errors.

Since stability results for many common schemes for approximating the wave equation $u_t = au_x$ and the heat equation $u_t = bu_{xx}$ are well-known [11], an often used practical strategy is to take the more restrictive of the two stability constraints for the wave and heat equations as the stability condition for the full advection-diffusion equation (1). However, the stability results for schemes approximating the equation $u_t = au_x + bu_{xx}$ cannot always be inferred from those for the wave and heat equations. Moreover, there is a danger of arriving at a condition that is more restrictive than necessary. For example, it is well known that Euler's method for the wave equation is *unconditionally unstable*, but the scheme applied to the advection-diffusion equation (Scheme E2E2 in Section 4) is actually *conditionally stable.* Worse yet, one can easily arrive at a condition that is not sufficient. For example, the stability condition of the scheme that consists of the Leap-Frog method applied to the u_x term and Euler's method applied to the u_{xx} term (Scheme LF2E2 in Section 4) is actually more restrictive than those of the corresponding methods applied to the wave and heat equations separately.

The definition of stability that we employ here is a generalization of the classical von Neumann stability condition and is designed to guarantee that the computed solution inherits one important property of the exact solution: that its norm remains bounded. We used a unified approach for deriving the stability results which is based on the Schur-Cohn theory of locating zeros of polynomials in terms of their coefficients. We apply this technique to analyse a collection of commonly used finite difference schemes that includes higher order approximations in both space and time.

Stability analysis for difference approximations to time dependent partial differential equations is often tricky, tedious and difficult. In this regard, it may be of interest to point out here that we have found an erroneous stability result for the Euler scheme E2E2 given originally by Fromm

([3], p.365) and later quoted by Roache ([13], p.44) and another erroneous result given in Roache ([13], p.O1) for the LF2E2 scheme. The Schur-Cohn technique that we employ here, however, is extremely powerful and general (especially for analysing schemes that span more than two time levels) and can be used in a systematic way to derive stability results for schemes (not necessarily finite difference schemes) that are not analysed here.

Some of the results that we shall present here are well-known and can be found in books such as Richtmyer and Morton [11], Roache [13], Vichnevetsky and Bowles [17]. However, we believe that some of the results are new. In any case, we hope that the collection of stability results in this paper will prove to be a useful reference.

In Section 2, we review centered difference approximations for the advective and diffusive terms. The general framework of stability analysis and the Schur-Cohn theory will be presented in Section 3. Analysis and results for a collection of commonly used schemes will be given in Section 4.

2. Centered Difference Approximations

In this section, we collect for reference purpose, some well-known results concerning centered finite difference operators for approximating the terms u_x and u_{xx}. Define the translation operator:

$$T(h)u(x) = u(x+h),\ h > 0. \tag{2}$$

We can now define the following difference operators in terms of T(h):

$$D_+(h) = (T(h) - T(0)) / h, \tag{3}$$

$$D_-(h) = (T(0) - T(-h)) / h, \tag{4}$$

$$D_0(h) = (T(h) - T(-h)) / 2h \equiv (D_+ + D_-) / 2. \tag{5}$$

(Notation: When the argument of a difference operator is left out, it is understood to be h.)

Approximations for u_x and u_{xx} using centered differences are well-known and are contained in the following theorems:

Theorem 1: Formally, the first derivative $D_x \equiv \partial/\partial x$ has the following expansion:

$$D_x - D_0 \sum_{j=0}^{\infty} (-1)^j\ \sigma_j\ (h^2 D_+ D_- /4)^j, \tag{6}$$

where

$$\sigma_j = [\ (j!)^2\ 2^{2j}\ ] / (2j+1)!. \tag{7}$$

Proof: See Kreiss and Oliger [7], Fomberg [2] and Vichnevetsky and Bowles [17].

Theorem 2: Formally, the second derivative $D_{xx} \equiv \partial^2/\partial x^2$ has the following expansion:

$$D_{xx} = D_+D_- \sum_{j=0}^{\infty} (-1)^j \mu_j (h^2D_+D_-/4)^j, \tag{8}$$

where

$$\mu_j - [(j!)^2 2^{2j}] / [(2j+1)!(j+1)]. \tag{9}$$

Proof: See Swartz [16].

We shall denote the 2m-th order difference approximation for D_x and D_{xx} by A_{2m} and B_{2m} respectively .

Definition 3: For $m \geq 1$, define:

$$A_{2m} = D_0 \sum_{j=0}^{m-1} (-1)^j \sigma_j (h^2D_+D_-/4)^j,$$

$$B_{2m} = D_+D_- \sum_{j=0}^{m-1} (-1)^j \mu_j (h^2D_+D_-/4)^j,$$

For the stability analysis, we shall need the Fourier transforms of these operators. We shall define the Fourier transform of an operator A by

$$\tilde{A} \equiv (A e^{iqx}) / e^{iqx}, \tag{10}$$

where

$$q = 2\pi\omega.$$

By noting that

$$D_0 e^{iqx} = ((e^{i\theta}-e^{-i\theta})/(2h)) e^{iqx} = (i \sin\theta / h) e^{iqx}, \tag{11}$$

and

$$(h^2D_+D_-/4)e^{iqx} = ((e^{i\theta}-2+e^{-i\theta})/4) e^{iqx} = -(\sin^2(\theta/2)) e^{iqx}, \tag{12}$$

where

$$\theta = 2\pi\omega h, \tag{13}$$

we can easily derive the following:

$$\tilde{A}_{2m} = iq (\sin\theta / \theta) \sum_{j=0}^{m-1} \sigma_j (\sin^2(\theta/2))^j, \tag{14}$$

$$\tilde{B}_{2m} = - q^2 [\sin^2(\theta/2)/(\theta/2)^2] \sum_{j=0}^{m-1} \mu_j (\sin^2(\theta/2))^j. \tag{15}$$

We note that $\tilde{A}_j$ is always purely imaginary and $\tilde{B}_j$ is always real.

The coefficients σ_j and μ_j in (7) and (9) are tabulated for orders up to six (i.e. j = 0, 1, 2) in Table 2-l. For computational purposes, it is often more convenient to transform equations (6) and (8) into stencil forms as:

$$A_{2m} = [\sum_{j=-L}^{L} \chi_j T(jh)] / h, \tag{16}$$

Table 2-1: Values Of σ_j, μ_j

j	σ_j	μ_j
0	1	1
1	2/3	1/3
2	8/15	8/45

$$B_{2m} = [\sum_{j=-L}^{L} \chi_j T(jh)] / h^2. \tag{17}$$

The values of χ_j and L are tabulated in Tables 2-2 for orders up to six.

Table 2-2: Stencils of A_{2m} and B_{2m}

Values of χ_j for D_x ($\chi_{-j} = \chi_j$)

m	L	χ_0	χ_1	χ_2	χ_3
1	1	0	1/2	-	-
2	2	0	8/12	- 1/12	-
3	3	0	45/60	- 9/60	1/60

Values of χ_j for D_{xx} ($\chi_{-j} = \chi_j$)

m	L	χ_0	χ_1	χ_2	χ_3
1	1	- 2	1		-
2	2	- 30/12	16/12	- 1/12	-
3	3	- 490/180	270/180	- 27/180	2/180

3. Stability Analysis - General Discussions

3.1. Definition of Stability

We shall consider the following Cauchy problem for (1):

$$u_t = au_x + bu_{xx}, \quad 0 \leq x \leq 1,$$

$$u(0,t) = u(1,t). \tag{18}$$

$$u(x,0) = f(x) \equiv \sum_{\omega=-n}^{n} \tilde{f}(\omega) e^{iqx}.$$

The exact solution is given by:

$$u(x,t) = \sum_{\omega=-n}^{n} \tilde{f}(\omega) e^{s(\omega)\, t} e^{iqx}, \tag{19}$$

. where

$$s(\omega) = iaq - bq^2. \tag{20}$$

Thus, the wave with frequency w travels with speed a and decays at an exponential rate given by e^{-bq^2}.

We discretize the spatial interval by a uniform grid with mesh size $h = 1/(2n+1)$ and use k to denote the time step. The most general difference approximation to the system (18) is of the form :

$$\phi_{-1}\, v(x,t+k) = \sum_{j=0}^{p} \phi_j\, v(x,t-jk), \tag{21}$$

where the $\phi_j \equiv \phi_j(a,b,h,k)$, $j = -1,0,1,..,p$ are spatial difference operators:

$$\phi_j = \sum_{\nu=-m_j}^{M_j} r_\nu^j(\, a,b,h,k)\; T^\nu(h). \tag{22}$$

The difference scheme defined above has a stencil that spans $p+2$ time levels, and on the time level $t-jk$ ($j = -1,0,1,...p$), it spans the mesh points from $x-m_jh$ to $x+M_jh$. See Figure 3-1.

We shall assume that $(\phi_{-1})^{-1}$ always exists and is bounded, so that (21) can be solved for $v(x,t+k)$. This usually amounts to requiring the band matrix defined by the linear operator ϕ_{-1} to be nonsingular. Any reasonable difference approximation has this property. Also, in practice, initial values have to be supplied for the time levels $t = 0,k,...pk$. These values can be supplied by using one-step (two-level) schemes for starting, for example, but for our analysis we shall assume that these values are obtained from the exact solution $u(x,t)$.

We look for approximate solutions to (21) of the form

$$v(x,mk) = \sum_{\omega=-n}^{n} \tilde{f}(\omega)\, R^m(\theta)\, e^{iqx}. \tag{23}$$

Figure 3-l: Stencil of a General Difference Scheme

```
                           t
x-m_-1 h                   |                    x+M_-1 h
 --x-.-.-x---x-----x---x---x-.-.--x-------   t+k
 --x-.-.-x---x-----x---x---x-.-.--x-------   t
 ..................|.....................
 ..................|.....................
 ..................|.....................
 --a-----x---x-----x---x---x------x-------   t-pk
                   |
x-m_p h                             x+M_p h
              ---> x direction
```

Note that if m = 0 in (23), we shall end up with the correct initial function $v(x,0) = \sum_{\omega=-n}^{n} \tilde{f}(\omega)e^{iqx}$. It can be shown ([8], Ch. 9) that $v(x,mk)$ as given by (23) will satisfy the difference equation (21) if R(8) satisfies the *characteristic quation:*

$$\tilde{\phi}_{-1}R^{p+1} - \sum_{j=0}^{p} \tilde{\phi}_j R^{p-j} = 0, \tag{24}$$

where

$$\tilde{\phi}_j = \sum_{\nu=-m_j}^{M_j} \tau_\nu^j e^{iq\nu h} = \sum_{\nu=-m_j}^{M_j} \tau_\nu^j e^{i\nu\theta}, \tag{25}$$

is the Fourier transform of ϕ_j. $R(\theta)$ is usually called the *amplification factor* of the difference scheme.

The characteristic equation (24) is a polynomial equation of degree p+l in R and so has p+l roots. Only one of these, usually called the *principal root,* corresponds to the approximate solution $v(x,mk)$ that we want. The other p roots are usually called *spurious* roots. In practice, any error introduced in the computation will be propagated by all the spurious roots. Therefore, unless there are some restrictions on the spurious roots, these propagated errors may become unbounded and overwhelm the approximate solution that we seek. Since the exact solution u(x,t) to the system has a norm (or energy) that is decreasing with time (or at least not growing with time), it seems reasonable to ask the same from the approximate solution $v(x,t)$. This is what Richtmyer and Morton [11] referred *to* as the *practical stability criteria.* It usually turns out that this condition will be satisfied if we restrict the time step k appropriately.

The left hand side of (24) is usually called the *characteristic polynomial.* Its coefficients are functions of a, b, h, k and 8. In general, we can express the characteristic polynomial of a p+2 time level scheme as:

$$H(R) = a_0 + a_1 R + \ldots + a_{p+1} R^{p+1}. \tag{26}$$

We shall call the p+l roots of H(R) $R_1, R_2, \ldots, R_{p+1}$, with R_1 being the principal root. It is clear from *(23)* that a necessary condition for the computed solution v not to be growing is :

$$|R_j| \leq 1 \quad \forall\ j. \tag{27}$$

This is usually called the *von Neumann Stability Condition* and polynomials with property (2) *are* called *von Neumann Polynomials.*

The von Neumann Condition is also sufficient for non-growing solutions for all two time level ($p = 0$) difference schemes with only one dependent variable [11]. However, it is not sufficient in general. The insufficiency mainly arises from the fact that when $p > 0$, condition (27) does not exclude the case of multiple roots on the unit circle. Therefore we have to modify the von Neumann condition a little bit.

Definition 4: We shall call polynomials H(R) with the following property:

$$|R_j| < 1 \quad \forall\ j,$$

Schur Polynomials.

We shall call polynomials H(R) with the following property:

$$|R_j| \leq 1 \quad \forall\ j, \quad \text{and} \quad R_j \text{ distinct on } |R| - 1. \tag{28}$$

Simple von Neumann Polynomials.

Definition 5: We shall call a scheme *stable* if its characteristic polynomial is a Simple von Neumann Polynomial $\forall\ \theta \in [0, 2\pi]$.

Note that this definition of stability is necessary and sufficient for the computed solution to not have a growing norm. Since the roots R_i are functions of a, b, h, k and 0, the stability condition will impose a restriction on the range of values that the first four parameters can take.

The notion of stability defined here is analogous to the notion of zero-stability in the theory of difference methods for the initial value problem in ordinary differential equation ([9], p.33 and [6], p.412). Condition (28) is the so-called *root-condition* in that theory.

Notice that our definition of stability (that of non-growing solutions) is slightly different from the definition of stability used in Richtmyer and Morton [11] and in particular the discussion about the effects of lower order terms on the stability for the heat equation on p.195 of their book does not apply in our case. Their definition of stability allows growth in the solution and, for diffusion problems like $u_t = b\, u_{xx}$, stability is practically unaffected by lower order terms like $a\, u_x$.

3.2. The Schur-Cohn Theory

There is a whole theory, originating from Schur [14, 15] that deals with the class of Simple von Neumann Polynomials. This theory, an excellent exposition of which can be found in a paper by J. J. H. Miller [10], enables one to determine conditions on the coefficients of the characteristic polynomial for it to be Simple von Neumann. We shall present the main results of that theory here and shall refer the reader to the original papers for more details.

Given a polynomial

$$\phi(z) = a_0 + a_1 z + \ldots + a_\nu z^\nu \equiv \sum_{j=0}^{\nu} a_j z^j,$$

of degree ν ($a_\nu \neq 0$) and having no zero at the origin ($a_0 \neq 0$), (any given polynomial can be reduced to this case without losing information about the location of its zeros), one can associate with ϕ another polynomial ϕ^*, satisfying the same conditions, and defined by

$$\phi^*(z) = \sum_{j=0}^{\nu} \tilde{a}_{\nu-j} z^j,$$

where $\tilde{a}$ denotes the complex conjugate of a. The reduced polynomial ϕ_1 is defined by

$$\phi_1(z) = (\phi^*(0)\, \phi(z) - \phi(0)\, \phi^*(z)) / z. \tag{29}$$

The main results that we need are contained in the following two theorems:

Theorem 6: ϕ is a Schur Polynomial iff $|\phi^*(0)| > |\phi(0)|$ and ϕ_1 is a Schur Polynomial.

Theorem 7: ϕ is a Simple von Neumann Polynomial iff either $|\phi^*(0)| > |\phi(0)|$ and ϕ_1 is a Simple von Neumann Polynomial or $\phi_1 \equiv 0$ and ϕ' is a Schur Polynomial (ϕ' denotes the derivative of ϕ with respect to its dependent variable).

By repeated applications of the above two theorems, it is possible to reduce the question of whether a n-th degree polynomial is a Simple von Neumann Polynomial to that for a first degree polynomial, which can be solved solved more easily by analytical means. These results turn out to be very useful for determining stability limits of difference schemes, as compared to first finding the roots of the characteristic polynomial explicitly and then determining their absolute values. Furthermore, this last approach may not even be applicable for polynomials of higher degrees.

4. Stability Analysis - Specific Schemes

4.1. Some Commonly Used Schemes

In this section, we shall present the stability analysis and results for some commonly used difference schemes for solving the advection-diffusion equation. We shall adopt the following convention for naming the schemes.

Definition 8: The name for a scheme shall consist of four fields:

Scheme Name: A B C D

where AB is used to denote how the scheme treats the advective term au,, and CD is used to denote how the scheme treats the diffusive term bu_{xx}. A and C are used to denote the time discretitation method used for the au, term and the bu_{xx} term respectively. B and D are used to denote the order of the centered differencing used for the au_x and bu_{xx} terms respectively.

The following abbreviations will be used for the time discretizations:

E - Forward Euler (first order, two levels, explicit)
BE - Backward Euler (first order, two levels, implicit)
CN - Crank-Nicolson (second order, two levels, implicit)
LF - Leap-Frog (second order, three levels, explicit)
DF - DuFort-Frankel (second order, three levels, explicit)
BD - Backward Differencing (second order, three levels, implicit)
P4 - Pade (fourth order, two levels, implicit)

For example, the following scheme:

$$(v^{m+1} - v^m)/k = aD_0v^m + bD_+D_-v^m$$

will be denoted by E2E2 because Euler's method is used to discretize in time and the spatial approximations are second order.

We shall analyse the following classes of schemes: EnEj, BEnBEj, CNnCNj, P4nP4j, BDnBDj, LFnCNj, LFnEj and LFnDFj, where n and j are even nonzero integers. This set of schemes is by no means exhaustive but is intended to include most of the commonly used schemes. It includes schemes that are first order, second order and fourth order in time; schemes that use the same order of spatial approximation for both the au, and bu_{xx} terms and those that use different orders for the two terms; schemes that use the same temporal scheme for both terms and those use different temporal schemes for them; explicit schemes and implicit schemes; and finally two-level and three-level schemes.

We collect in Table 4-1 the exact definitions of these schemes and their stability conditions. For reference purpose, we have also indicated the order of the truncation errors for each scheme. For more details on the error analysis and the stencils for these schemes, the reader is referred to [1].

4.2. Stability Analysis

Next, we shall present the stability analysis for the numerical schemes presented in Table 4-1. We shall apply the basic Schur-Cohn theory presented in Section 3.2 to the characteristic polynomial of each of the schemes. Only the three-level schemes make non-trivial use of this theory and we shall present only their analysis in details.

We shall use the following definitions in this section:

Definition 9: Define:

$$\beta = 4kb/h^2 ,$$

$$\alpha = ak/h ,$$

$$\gamma = -\, bk\, \widetilde{B}_j ,$$

$$\delta = ak\, (\widetilde{A}_n/i).$$

We note that all the above quantities are real and β and γ are non-negative. The indices n and j should be clear from the context.

We shall also need the following definitions:

Definition 10: Define:

$$M = \min(n, j),$$

$$\kappa_n = \sum_{\nu=0}^{n/2-1} \mu_\nu ,$$

$$\rho_j = \sum_{\nu=0}^{j/2-1} \sigma_\nu ,$$

where the μ_ν's and the σ_ν's are defined in Section 2. Specifically, $\kappa_2 = 1$, $\kappa_4 = 5/3$ and $\kappa_6 = 11/5$; $\rho_2 = 1$, $\rho_4 = 4/3$ and $\rho_6 = 68/45$.

1) **EnEj**

The amplification factor is given by

$$R = 1 + i\delta - \gamma.$$

We have to find conditions on h and k so that $|R| \leq 1$ for $\theta \in [-\pi, \pi]$. This leads to the

Table 4-1: Summary of Stability Results for Schemes for $u_t = au_x + bu_{xx}$

Notation:
- h: space step, k: time step.
- n, j : positive even integers., $M = \min[n, j]$.
- A_n : n-th order centered difference operator for u_x.
- B_j : j-th order centered difference operator for u_{xx}.
- Order(p, q) : Truncation error $= O(h^p) + O(k^q)$.

Scheme	Definition	Stability Condition (order in x, order in t)
E2E2	$(v^{m+1} - v^m)/k = (aA_2 + bB_2)v^m$	$k < \min[2b/a^2, h^2/2b]$ order(2, 2)
BEnBEj	$(v^{m+1} - v^m)/k = (aA_n + bB_j)v^{m+1}$	Unconditionally Stable order(M, 1)
CNnCNj	$(v^{m+1} - v^m)/k =$ $(aA_n + bB_j)(v^{m+1} + v^m)/2$	Unconditionally Stable order(M, 2)
P4nP4j	$(I - G/2 + G^2/12)v^{m+1} =$ $(I + G/2 + G^2/12)v^m$ where $G = k(aA_n + bB_j)$	Unconditionally Stable order(M, 4)
BDnBDj	$(3/2)(v^{m+1}-v^m)/k - (1/2)(v^m-v^{m-1})/k$ $= (aA_n + bB_j)v^{m+1}$	Unconditionally Stable order(M, 2)
LFnCNj	$(v^{m+1} - v^{m-1})/(2k) = aA_n v^m$ $+ bB_j(v^{m+1} + v^{m-1})/2$	Same as that for LFn: n = 2 : $k < h/\|a\|$ n = 4 : $k < 0.7287\ h/\|a\|$ n = 6 : $k < 0.6305\ h/\|a\|$ order(M, 2)
LF2E2	$(v^{m+1} - v^{m-1})/(2k) = aA_2 v^m + bB_2 v^{m-1}$	$(ak/h)^2 + (4bk/h^2) < 1$ order(2, 2)
LFnDFj	$(v^{m+1}-v^{m-1})/(2k) = aA_n v^m + bB_j m$ $- (\eta_j b/h^2)(v^{m+1} - 2v^m + v^{m-1})$ $\eta_2 = 1$ $\eta_4 = 413$ $\eta_6 = 68/45$	n=2, j=2: $k < h/\|a\|$ n=4, j=2: $k < 0.5311\ h/\|a\|$ n=2, j=4: $k < 0.9685\ h/\|a\|$ n=4, j=4: $k < 0.5453\ h/\|a\|$ Error $= O(h^M, k^2, \eta_j (k/h)^2)$

condition:

$$(1 - \gamma)^2 + \delta^2 \leq 1.$$

It follows that two *necessary* conditions are:

$$\gamma \leq 2 \quad \text{and} \quad |\delta| \leq 1,$$

which reduces to

$$\beta\rho_j/2 \leq 1 \quad \text{and} \quad |\alpha|\kappa_n \leq 1.$$

Sufficient conditions, however, are more difficult to derive. analytically, especially for larger values of n and j. For the simplest case of n = j - 2, it can be shown that the necessary and sufficient conditions are:

$$\beta \leq 2 \quad \text{and} \quad \alpha^2 \leq \beta/2,$$

which reduces to (30)

$$k < \min(2b/a^2, h^2/2b).$$

The analytical solution of this problem is not difficult but a bit tedious and can be found in [1]. For a geometric proof, see [12]. Results for the general case are not known.

Remarks: The stability of the method E2E2 was studied by Roache [13] and a two-dimensional version by Fromm [3]. Instead of condition (30), they found lower bounds on the spatial step size h independent of the temporal step size k, the so-called cell Reynolds Number limitation, which is more restrictive. Our results show that h can be as small as we wish. As long as k is small enough, the scheme is stable. See also Hirt [5].

2) **BEnBEj**

The amplification factor is:

$$R = 1 / (1 - i\delta + \gamma).$$

It follows from the definitions of $\tilde{A}_n$ and $\tilde{B}_j$ in Section 2 that

$$|R|^2 = 1 / (1 + \gamma)^2 + \delta^2 \leq 1.$$

Hence this scheme is unconditionally stable.

3) **CNnCNj**

The amplification factor is:

$$R = (1 + i\delta/2 - \gamma/2) / (1 - i\delta/2 + \gamma/2).$$

It follows that

$$|R|^2 = ((1-\gamma/2)^2 + (\delta/2)^2) / ((1+\gamma/2)^2 + (\delta/2)^2) \leq 1.$$

Hence this scheme is unconditionally stable.

4) **P4nP4j**

The amplification factor is:

$$R = (1 + \tilde{G}/2 + \tilde{G}^2/12) \,/\, (1 - \tilde{G}/2 + \tilde{G}^2/12).$$

Let $\tilde{G} = \mathcal{R} + i\mathcal{I}$ where $\mathcal{R}$ and $\mathcal{I}$ are real. Then it follows that

$$|R|^2 = ((1-\gamma/2+\mathcal{R})^2 + (\delta/2+\mathcal{I})^2) \,/\, ((1+\gamma/2+\mathcal{R})^2 + (\delta/2+\mathcal{I})^2) \leq 1.$$

Hence this scheme is unconditionally stable.

5) **BDnBDj**

In the notation developed in Section 3.2, the characteristic polynomial is given by:

$$\phi(z) = (3/2 + 7 - i\delta)z^2 - 2z + 1/2 ,$$

and

$$\phi^*(z) = 1/2z^2 - 2z + (3/2 + 7 + i\,\delta) .$$

We shall use Theorem 7 to show that $\phi(z)$ is a simple von Neumann polynomial. It would then follow that the scheme is stable.

The condition $|\phi^*(0)| > |\phi(0)|$ is certainly always satisfied. We next compute ϕ_1 as

$$\phi_1(z) = [(3/2 + \gamma)^2 + \delta^2 - 1/4]z - 2(1 + 7 \quad i\delta) .$$

q+(z) is simple von Neumann iff

$$|2(1 +\gamma + i\delta)|^2 \leq [(3/2 + \gamma)^2 + \delta^2 - 1/4]^2 .$$

This can easily he shown to be true for any real 7 and δ. Thus $\phi(z)$ is simple von Neumann and the scheme is unconditionally stable.

6) **LFnCNj**

The characteristic polynomial is given by:

$$\phi(z) = (1 + \gamma)z^2 - 2i\delta z - (1 - 7).$$

We thus get:

$$\phi^*(z) = (7 - 1)z^2 + 2i\delta z + (1 + \gamma).$$

The condition $|\phi^*(0)| > |\phi(0)|$ reduces to $|1 + 7 > |1 - \gamma|$, which is always true because 7 is positive.

Next we compute $\phi_1(z)$ as

$$\phi_1(z) = 4\gamma z - 4i\gamma\delta .$$

$\phi_1(z)$ is simple von Neumann iff

$$|\delta| < \; . \tag{31}$$

Note that condition (31) is the same as the stability criterion for the Leap-Frog scheme applied to $u_t = au_x$, with n-th order centered differencing in space. The criteria are all of the form $|ak/h| \leq c_n$, where the constants c_n can be found in Fomberg [2], for example. In particular, for $c_2 = 1$, $c_4 = 0.7287$ and $c_6 = 0.6305$.

7) **LFnEj**

The characteristic polynomial is given by:

$$\phi(z) = z^2 - 2i\delta z - (1 - 27).$$

We thus get:

$$\phi^*(z) = (2\gamma - 1)z^2 + 2i\delta z + 1.$$

Now the condition $|\phi^*(0)| > |\phi(0)|$ reduces to $1 > |1 - 2\gamma|$ which will be satisfied iff

$$\gamma < 1 \;,$$

which reduces to

$$\beta\rho_j < 1.$$

This is the same as the stability condition for Euler's method applied to the u_{xx} term with *a* time step of 2k and is clearly a necessary condition for stability.

We next compute $\phi_1(z)$ as

$$\phi_1(z) = z(1 - (1 - 2\gamma)^2) - 4i\delta\gamma.$$

$\phi_1(z)$ will be simple von Neumann iff

$$|4\delta\gamma| \leq 1 - (1-2\gamma)^2,$$

which reduces to

$$|\delta| \leq 1 - 7.$$

This is a relationship involving β, α and θ and we want to derive conditions involving β and α for it to hold for all values of θ. In the case of $n = 2$ and $j = 2$, this has been worked out in [1] and the necessary and sufficient condition is:

$$|\alpha|^2 + \beta \leq 1. \tag{32}$$

Results for more general values of n and j are not known.

Remark: Roache [13] considered this scheme for $n = j = 2$, but he claimed that the stability analysis of the advection and diffusion terms may be analyzed separately, and thus obtaining the conditions $|\alpha| \leq 1$ and $\beta \leq 1$. (In his book, he had the equivalent of $\beta \leq 2$, which I believe is either a typo or a mere oversight.) These two separate conditions are much less restrictive than (32) and we believe that Roache's results were erroneous. Rigal [12] derived the correct stability limit using a different approach but he did not show that it is both necessary and sufficient.

8) LFnDFj

Generalized Du Fort-Frankel methods have been studied by Gottlieb and Gustafsson [4]. When adapted to our case for the equation $u_t = au_x + bu_{xx}$, the scheme LFnDFj becomes

$$(v^{m+1} - v^{m-1})/2k = (aA_n + bB_j)v^m - (\eta_j b/h^2)(v^{m+1} - 2v^m + v^{m-1})$$

where η_j is a positive constant chosen to make the scheme unconditionally stable for the case a = 0 (the heat equation). The conditions is exactly:

$$\eta_j \geq \rho_j . \tag{33}$$

The truncation error of the scheme is $O(h^M, k^2, \eta_j(k/h)^2)$. Thus the larger the value of η_j is, the larger is the truncation error. Therefore, in what follows, we shall assume that η_j takes on the value of the lower bounds given in (33).

The characteristic polynomial is:

$$\phi(z) = z^2(1 + \eta_j\beta/2) - z(\eta_j\beta + 2i\delta - 2\gamma) - (1 - \eta_j\beta/2) .$$

It follows that:

$$\phi(z) = z^2(-1 + \eta_j\beta/2) - z(\eta_j\beta - 2i\delta - 2\gamma) + (1 - \eta_j\beta/2) .$$

The condition $|\phi^*(0)| > |\phi(0)|$ reduces to

$$|1 + \eta_j\beta/2| > |1 - \eta_j\beta/2| ,$$

which is always satisfied because η_j and β are both positive.

We compute $\phi_1(z)$ as:

$$\phi_1(z) = 2\eta_j\beta z - 2(\eta_j\beta - 2\gamma + \eta_j\beta i\delta) .$$

Hence the stability criterion is given by

$$|1 + 2\gamma/(\beta\eta_j) + i\delta| \leq 1 ,$$

which can be written as:

$$\alpha^2 \leq (1 - (1 + \tilde{B}_j(h^2/2\eta_j))^2)/|\tilde{A}_n h|^2 . \tag{34}$$

It can easily be verified that, for a given n and j, the right hand side of (34) is a function of θ only, and thus its minimum can be found, at least numerically, to yield au upper bound for α^2 as the stability criterion for the scheme LFnDFj. Moreover, it can also be seen from the form of (34) that the limitation on k is more restrictive than the corresponding limitation for the Leap

Frog method applied to $u_t = au_x$. The upper bounds have been computed numerically and their values are given in Table 4-2.

Table 4-2: Stability Constants for Scheme LFnDFj

Stability Condition : $\alpha^2 \le c_{n,j}$, where $c_{n,j}$ is given below:

n \ j	2	4	6
2	1.0	0.9685	0.9242
4	0.5310	0.5458	0.5391
6	0.3976	0.4231	0.4281

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