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FINITE-DIFFERENCE METHODS FOR SINGULAR PERTURBATION AND NAVIER-STOKES PROBLEMS

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Abstract. The linear equation $\epsilon u_{xx} + xu_x = 0$, $0 < x < 1$, is proposed as a model for investigating interesting features of the behavior of difference methods for realistic multidimensional nonlinear elliptic problems, especially Navier-Stokes problems. We give an analytic and experimental comparison of several difference schemes for this model problem. An unusual scheme for the Navier-Stokes equations is suggested by these results. An experiment shows that this scheme performs better than a more obvious one.

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1. Introduction.

This report attempts to elucidate some of the interesting and poorly understood phenomena that have been observed when solving steady nonlinear problems, including Navier-Stokes problems,-.by finite-element and finite-difference methods. The phenomena in question include unwarranted oscillations (especially of derivatives of the solution) which are most severe in boundary layers, a relationship between accuracy and a cell Reynolds number, a relationship between difficulty in obtaining solutions to the discrete equations and a cell Reynolds number, and spurious multiple solutions of the discrete equations. We find that difference schemes for the linear two-point boundary value problem

$$Lu \equiv \varepsilon u_{xx} + x u_x = 0 \tag{1a}$$

$$u(0) = 1 \,,\; u(1) = 0 \tag{1b}$$

exhibit similar features and offer possible explanations.

Examples of difficulties with discretitations of fluids problems abound. They include nonphysical solutions ([1], [5], [7]) and failure of the given methods to find any solution [6]. The second problem is often the result of a limit point in the solution curve of the discrete equations; an example is given in Section 4 of this paper.

We consider four difference schemes for (1): an upwind scheme, the standard, centered, second-order scheme, and two other centered second-order schemes. We give extensive numerical results, and also analyze their behavior as $\varepsilon \to 0$ for fixed mesh spacing h. We find that the standard centered scheme has solutions which grow without bound; a condition $h/\sqrt{\varepsilon} <$ const. is required to bound them. The form of this condition is reasonable: the solution to (1), $u(x,\varepsilon) = 1 - \mathrm{erf}(x/\sqrt{2\varepsilon})/\mathrm{erf}(1/\sqrt{2\varepsilon})$ varies rapidly in a boundary layer of thickness $O(\sqrt{\varepsilon})$. All the other schemes yield bounded solutions for fixed h and small ε. One of them, however, appears to be more accurate.

Abrahamsson, Kreiss, and Keller [3] investigated difference methods for $\varepsilon u_{xx} + Cu_x = 0$. For this problem the boundary layer thickness is O(e), and a cell Reynolds number restriction $h/s \leq$ const is necessary for a nonoscillatory solution.

A more obvious one-dimensional model for Navier-Stokes is the steady Burgers' equation $\varepsilon u_{xx} - uu_x = 0$. (Our concern with multiple solutions, a non-linear phenomenon, argues for this model.) Kellogg, Shubin, and Stephens [1] showed that any reasonable three-point second-order-accurate centered scheme leads to multiple solutions unless a condition $h/c \leq$ const is imposed. The solution has boundary layers $O(\varepsilon)$ in thickness, so this isn't surprising. They also showed that an $O(h)$ upwind scheme can give spurious multiple solutions, which is.

We do not believe that these pessimistic results hold for Navier-Stokes problems. First, boundary layers in Navier-Stokes are $O(\sqrt{\varepsilon})$ thick (ε^{-1} = Reynolds number) [2]. Second, we give numerical results in Section 4 for the driven cavity flow that do not exhibit these problems. We use two second-order, centered schemes. One seems to give a unique solution for $h/\sqrt{\varepsilon} \leq$ const. The other appears to have a unique solution for all h and ε

and is more accurate than the first. This nonobvious scheme was suggested by the best of the schemes for (1).

Some notation for finite differences will be useful. Let $h > 0$ be given. For a function $u(x)$, let

$$\begin{aligned} D_+u(x) &\equiv (u(x+h) - u(x))/h \\ D_-u(x) &\equiv (u(x) - u(x-h))/h \\ D_+D_-u(x) &\equiv (u(x+h) - 2u(x) + u(x-h))/h^2 \\ D_0u(x) &\equiv (u(x+h) - u(x-h))/2h \ . \end{aligned}$$

We shall consider schemes that approximate $u(x_i)$ for a uniform grid $x_i = ih$, $0 \leq i < n + 1$, where $h = 1/(n+1)$.

2. Difference Schemes for the Model Problem.

We consider four schemes for (1): an upwind scheme, two centered schemes of "convective" type, and a centered scheme of "divergence" type. All four treat the term uxx the same way:

$$\varepsilon u_{xx} \approx \varepsilon D_+ D_- u(x) \quad . \tag{3}$$

The schemes differ in their approximation to xu_x:

Scheme **1**.

$$xu_x(x) \approx xD_+u(x) \quad . \tag{4a}$$

Scheme 2.

$$xu_x(x) \approx xD_0u(x) \quad . \tag{4b}$$

Scheme 3.

$$xu_x(x) \approx (x+\frac{h}{2})D_+u(x) + (x-\frac{h}{2})D_-u(x) \quad . \tag{4c}$$

Scheme 4.

$$xu_x(x) \approx D_0(xu)(x) - u(x) \quad . \tag{4d}$$

Scheme 3 can be derived by applying Galerkin's method to a variational form of (1) using continuous piecewise - linear approximations, the usual "hat function" basis, and one-point Gauss quadrature (the midpoint rule) to evaluate the integrals. Scheme 4 is based on the identity $xu_x = (xu)_x - u$.

The three centered schemes are closely related. In fact, all lead to tridiagonal systems

$$T_s\underline{u} = \underline{f} \ , \ s = 1,2,3,4 \ ,$$

where u_i approximates $u(x_i)$, $1 \leq i \leq n$. (We take $u_0 = 1$ and $u_{n+1} = 0$ as boundary conditions.) Let

$$T_s = \varepsilon A + B_s \quad , \tag{5}$$

where

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix}$$

is due to the approximation for εu_{xx}, and B_s represents the approximation to xu_x. Then

(6a) $$B_3 = B_2 + \frac{h^2}{4} A$$

and

(6b) $$B_4 = B_2 + \frac{h^2}{2} A \quad .$$

First, examine the actual behavior of the schemes. Figures 1-8 exhibit their error as a function of ε and n , for $\varepsilon = 1, 10^{-2}, 10^{-4}$, and 10^{-6}, and $n = 16, 32, \ldots,$ 1024. The discrete ℓ^∞ - norm

$$\|e\|_\infty = \max_{0 \le i \le n} |e(x_i)|$$

and ℓ^2 - norm

$$\|e\|_2 = \left(\sum_{i=0}^{n} (e(x_i))^2\right)^{\frac{1}{2}}$$

are shown. Several points should be emphasized. These discrete error measures give no information about the quality of approximation in the boundary layer when $\sqrt{\varepsilon} \ll h$. The apparent growth of the error with decreasing h in Figures 1,4,5, and 8 is an artifact of this effect. For grids sufficiently fine to resolve the boundary layer, observe that the upwind scheme is not competitive with the centered schemes, and that scheme 3 is substantially more accurate than either of the other centered schemes.

When $\varepsilon \to 0$ for fixed h, or, equivalently, as $h \to 1$ for fixed ε, the schemes behave differently. Scheme 1 and 4 find the "outer solution*' $u \approx 0$. Scheme 2 apparently "blows-up" as $\varepsilon \to 0$. Scheme 3 doesn't get the outer solution, but neither does the error blow-up. Rather, it approaches a bound which depends on n, but not ε.

2.1 Properties of the differential equation.

A variational form of problem (1) is obtained in this section. The existence and uniqueness of solutions of (1) follow from the Lax-Milgram theorem using the H_0^1-ellipticity of a bilinear form. This, in turn, holds because the symmetric part $M = (L+L^*)/2$ of the operator L 'is just $\varepsilon\partial_x^2 - \frac{1}{2}$, which is H&elliptic.

Let H^1 be the space of continuous functions on $[0,1]$ with square-integrable first derivative. Let $H_0^1 \equiv \{u\varepsilon H^1 | u(0) = u(1) = 0\}$, and $H_E^1 \equiv \{u\varepsilon H^1 | u(0) = 1,\ u(1) = 0\}$. We associate with (I) the problem

(VI): Find $u\varepsilon H_E^1$ such that

$$a(u,v) \equiv \int_0^1 (-\varepsilon u_x v_x + x u_x v)dx = 0$$

for all $v\varepsilon H_0^1$.

Let u_0 be an arbitrary element of H_E^1 (for example, $u_0(x) = 1-x$). Then $H_E^1 \equiv \{w+u_0 | w\varepsilon H_0^1\}$. Thus, (V1) is equivalent to

(V2): Find $w\varepsilon H_0^1$ such that

$$a(w + u_0, v) = 0$$

. for all $v\varepsilon H_0^1$. Equivalently, since a(,) is bilinear,

$$a(w,v) = - a(u_0,v)$$

for all $v\varepsilon H_0^1$.

Define norms

$$||u||_0^2 \equiv \int_0^1 u^2(x)dx \ , \ u\varepsilon L^2$$

and

$$||u||_1^2 \equiv \int_0^1 (u_x^2(x) + u^2(x))dx \ ,$$

$u\varepsilon H^1$. Note that the mapping- $v \rightarrow a(u_0,v)$ is a bounded linear functional on H^1:

by the Cauchy-Schwartz inequality,

$$\text{(7)} \qquad |a(u_0,v)| \le \varepsilon \|u_{0x}\|_1 \|v\|_1 + \|u_0\|_1 \|v\|_0 \le C\|v\|_1 .$$

Also, for $v \varepsilon H_0^1$,

$$-a(v,v) = \varepsilon \|v_x\|_0^2 - \int_0^1 (x v_x v) dx = \varepsilon \|v_x\|_0^2 + \tfrac{1}{2} \|v\|_0^2 \ge \frac{\varepsilon}{1+\pi^{-2}} \|v\|_1^2$$

where we have integrated by parts and used the Rayleigh-Ritz inequality ([9]). This lower bound on a(v,v), together with **(7),** allows the Lax-Milgram theorem ([13]) to be used, showing that (V2) has a unique solution. The key is that for smooth $v (v \varepsilon H^2 \cap H_0^1)$,

$$a(v,v) = \int_0^1 (\varepsilon v_{xx} + x v_x) v \, dx = \int_0^1 (Mv) v \, dx$$

where

$$\text{(8)} \qquad M: v \to \varepsilon v_{xx} - \tfrac{1}{2} v$$

is the symmetric part of the operator L.

We call M the **symmetric** part of L because if L = M + N, then M is self-adjoint: for all $u, v \varepsilon H_0^1$,

$$\int_0^1 Mu \, v \, dx = \int_0^1 u \, Mv dx$$

(just integrate by parts), and $N: u \to x u_x + \tfrac{1}{2} u$ is skew-adjoint: for all $u, v \varepsilon H_0^1$

$$\int_0^1 Nu \, v \, dx = - \int_0^1 u \, Nv \, dx .$$

Such a decomposition is unique.

3. An analysis of the methods.

In this section we consider two questions about each scheme:

Question 1.

Is the symmetric part,

$$M_s(h,\varepsilon) \equiv \tfrac{1}{2}\left[T_s + T_s^T\right],$$

negative definite? In otherwords, does a bound

$$M_s(h,\varepsilon) \leq \delta(h) < 0$$

hold uniformly as $\varepsilon \to 0$?

Question 2.

Does there exist a bound

$$\|T_s^{-1}\| \leq K(h)$$

uniform as $\varepsilon \to 0$?

The significance of question 2 is clear - bounds on $\|T^{-1}\|$ are an essential part of the error estimates of all the schemes.

Question 1 is motivated by the consideration of nonlinear problems. Suppose a nonlinear equation $F(u,\varepsilon) = 0$ has a "basic" solution $u_0(\varepsilon)$ such that the linear problem $F_u\left[u_0(\varepsilon),\varepsilon\right] v = 0$ has only the trivial solution for all $\varepsilon > 0$. Then we can "continue" the solution $u_0(\varepsilon)$ to arbitrarily small values of ε without any bifurcation or limit points occuring. A sufficient condition for this is that the symmetric part of the operator $F_u\left[u_0(\varepsilon),\varepsilon\right]$ be negative (or positive) definite in some appropriate sense - for a second-order differential operator with Birichletboundary conditions, the associated bilinear form should be H&elliptic.

Now suppose the original problem is to be approximated by a family of discrete problems $F^h(u,\varepsilon) = 0$. Then it is desirable that for h sufficiently small these problems possess solutions $u_0^h(\varepsilon)$ that are (locally) unique as $\varepsilon \to 0$. Again, the invertibility of the Jacobian matrices $F_u^h[u_0^h(\varepsilon),\varepsilon]$ suffices for existence and local uniqueness, and definiteness of the symmetric part of F_u^h suffices for invertibility. Moreover, uniform definiteness precludes the possibility that the solution blows up as $\varepsilon \to 0$, is essential if the condition number of the **Jacobians** is to be bounded, and is desirable if iterative methods will be used to solve the discrete problems.

Here are the answers to question 1. In each case, $T_s = \varepsilon A + B_s$, so

$$M_s = \varepsilon A + C_s ,$$

where

$$C_s \equiv \tfrac{1}{2}(B_s + B_s^T) .$$

Scheme 1.

$$B_1 = \begin{bmatrix} -1 & 1 & & & \bigcirc \\ & -2 & 2 & & \\ & & \ddots & \ddots & \\ & & & & n-1 \\ \bigcirc & & & & -n \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -1 & \tfrac{1}{2} & & & & & \bigcirc \\ \tfrac{1}{2} & \ddots & \ddots & & & & \\ & (\tfrac{j-1}{2}) & -j & j/2 & & & \\ & & & \ddots & \ddots & (\tfrac{n-1}{2}) & \\ \bigcirc & & & & (\tfrac{n-1}{2}) & -n & \end{bmatrix}$$

By **the** Gershgorin theorem, $C_1 \leq -\tfrac{1}{2}$, so $M_1 \leq -\tfrac{1}{2}$. (A **is negative definite.)**

Scheme 2.

$$C_2 = \tfrac{1}{4}\begin{bmatrix} O & -1 & & O \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ O & & -1 & O \end{bmatrix} \equiv -\frac{h^2}{4}A - \tfrac{1}{2}I .$$

Therefore, $M_2 = (\varepsilon - \frac{h^2}{4})A - \frac{1}{2}I$. This is negative definite for h sufficiently small; in other words we have a "cell Reynolds number" condition for negative definiteness. In fact, if $\varepsilon > h^2/4$, then $M_s < -\frac{1}{2}$. If $(\varepsilon - h^2/4) < 0$, then definiteness depends on whether or not $(\varepsilon - h^2/4)\lambda_1 > \frac{1}{2}$, where λ_1 is the (negative) eigenvalue of A of largest magnitude. It is well known that $\lambda_1 = \frac{-4}{h^2}(\sin \frac{n-1}{2n}\pi)^2$, so the condition for definiteness is this. If

$$\varepsilon \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \frac{h^2}{8}\left(2 - \frac{1}{(\sin \frac{(n-1)\pi}{2n})^2}\right) \sim \frac{h^2}{8}$$

then

$$M_2 \text{ is } \begin{cases} \text{negative definite} \\ \text{negative semidefinite} \\ \text{indefinite} \end{cases}$$

Scheme 3.

$$C_3 = -\tfrac{1}{2}I .$$

Thus, this scheme gets the symmetric part exactly right, in view of (8). M_3 is uniformly negative, and its eigenvalues approach $-\frac{1}{2}$ as $\varepsilon \to 0$.

Scheme 4.

$$C_4 = \frac{h^2}{4}A - \tfrac{1}{2}I .$$

Again, the symmetric part is uniformly negative definite. For small ε, however, M_4 is a poor approximation to the operator M.

Here are the answers to question 2. In each case the question is addressed by obtaining an upper bound on $\|T^{-1}\|_2$, the spectral norm of T^{-1}, or showing that none exists. Of course,

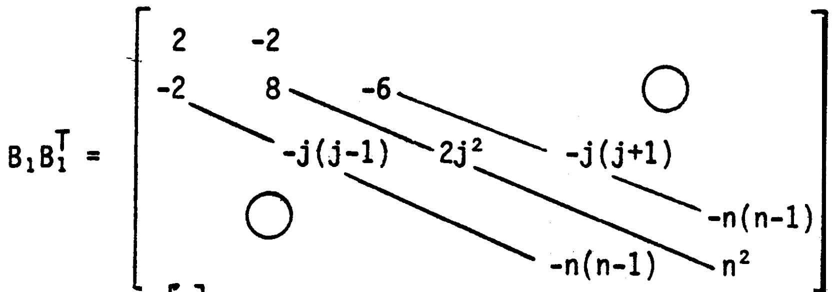
$$\|T^{-1}\|_2 = 1/\inf_{\|x\|_2=1} \|Tx\|_2 ,$$

where, for $x = (x_1, x_2, \ldots, x_n)^T$

$$\|x\|_2^2 \equiv \sum x_i^2 .$$

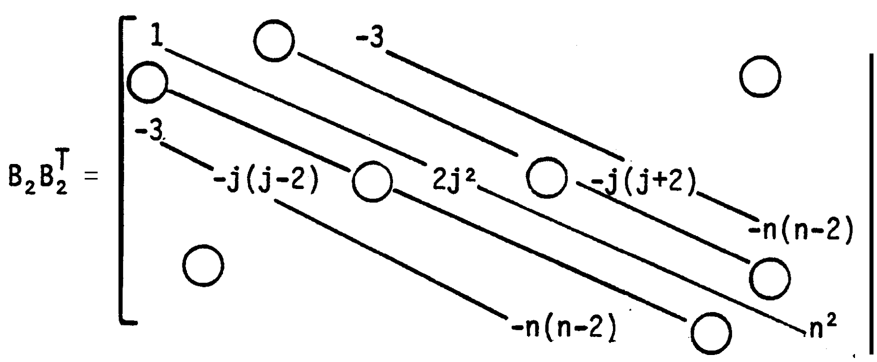
Moreover, $\|T^{-1}\|_2 = 1/\sigma_1$ where $0 < \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n$ are the singular values of T.* Finally, since $T = \varepsilon A + B$, the singular values of T converge, as $\varepsilon \to 0$, to those of B. Thus, question 2 is answered in the affirmative if a lower bound on the smallest singular value of B can be obtained.

Scheme 1.



The Taussky theorem [8] states that if an eigenvalue lies on the boundary of the union of the Gershgorin disks, then it is a point of intersection of all the disks. Since the disk of the last row of $B_1B_1^T$ doesn't include zero, and zero is on the boundary of the other disks, $B_1B_1^T$ is positive definite.

Scheme 2.



Permute rows and columns so that odd numbered equations and unknowns precede all

* The singular values of a square matrix A are the positive roots of the eigenvalues of A^TA, which are the same as those of AA^T.

even numbered equations and unknowns. The resulting permuted matrix has the form

$$PB_2B_2^TP^T = \left[\begin{array}{c|c} B_{11} & \bigcirc \\ \hline \bigcirc & B_{22} \end{array}\right] .$$

Since all row sums of B_{22} are nonnegative and the last strictly positive, B_{22} is positive definite (by the Taussky theorem). As for B_{11}, when n is odd it is singular and when n is even it is positive definite. To see this, note that when n is odd,

$$B_{11} = \begin{bmatrix} 1 & -3 & & & \\ -3 & 18 & \ddots & & \\ & -j(j-2) & 2j^2 & -j(j+2) & \\ & & \ddots & \ddots & -n(n-2) \\ & & & -n(n-2) & n^2 \end{bmatrix}$$

and $B_{11}\,\underline{z} = 0$ where $\underline{z} = (n, n/3, n/5, \ldots, n/(n-2), 1)^T$. When n is even, $|B^{(j)}| = \prod_{\ell=0}^{(j-1)/2} (j-2\ell)^2$, $j = 1,3, \ldots, n-1$, where $|B^{(j)}|$ is the determinant of the leading principal submatrix of order $(j+1)/2$. The cases $j = 1,3$ are trivial. For $j \geq 5$, expand by cofactors to show that

$$|B^{(j+2)}| = 2(j+2)^2|B^{(j)}| - (j(j-2))^2|B^{(j-2)}|$$
$$= ((j+2)\cdot j\cdot(j-2)\ldots 1)^2 ,$$

using induction on j. Since $B^{(n-1)} = B_{11}$ has positive determinant and cannot have a negative eigenvalue, it is positive definite.

Scheme 3.

$(B_3 + \frac{1}{2} I)$ is the skew-symmetric matrix

$$N_3 \equiv \frac{1}{4} \begin{bmatrix} \bigcirc & 3 & & \bigcirc \\ -3 & \ddots & 5 \;\ddots & \\ -5 \;\ddots & \ddots & \ddots & 2n-1 \\ \bigcirc & & -(2n-1) & \bigcirc \end{bmatrix} .$$

Since

is bounded uniformly in ε and h.

Scheme 4 (cdd).

$$B_4B_4^T = \frac{1}{4}\begin{bmatrix} 8 & -2 & -4 & & & & \\ -2 & 14 & -2 & -9 & & & \\ -4 & \ddots & \ddots & \ddots & \ddots & & \\ & -(j^2-2j+1) & -2 & 2j^2+6 & -2 & -(j^2+2j+1) & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots & -2 \\ & 0 & & & -(n^2-2n+1) & -2 & n^2+4 \end{bmatrix}$$

Again, the Taussky theorem shows that $B_4B_4^T$ is positive definite.

4. Navier-Stokes Equations.

The driven cavity problem, a standard test problem in numerical fluid dynamics, is to solve the equations governing the flow of a viscous uncompressible fluid in a square, two-dimensional box with a wall that slides over the fluid. Figure 9 shows the flow and the boundary conditions.

The equations are

(8a) $$-\Delta\Psi = \omega, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

(8b) $$\varepsilon\Delta\omega = u\omega_x + v\omega_y$$

where Ψ is the streamfunction, ω the vorticity, and ε the kinematic viscosity. $\binom{u}{v}$ is the velocity field,

$$\underset{\sim}{u} \equiv \begin{pmatrix} \Psi_y \\ -\Psi_x \end{pmatrix} .$$

The vorticity can be eliminated from (8b) using (8a); a single equation for the streamfunction,

$$\Delta^2\Psi - \frac{1}{\varepsilon}(\Psi_y\Delta\Psi_x - \Psi_x\Delta\Psi_y) = 0 .$$

The appropriate boundary conditions are shown in figure 9.

A standard finite-difference scheme for (8) is (assuming as uniform grid)

$$-\Delta_h \Psi = \omega$$

$$\varepsilon\Delta_h\omega = D_{0y}\Psi D_{0x}\omega - D_{0x}\Psi D_{0y}\omega ,$$

where $D_{0x}\Psi = (\Psi(x+h,y) - \Psi(x-h,y))/2h$,

$$D_{0y}\Psi = (\Psi(x,y+h) - \Psi(x,y-h))/2h ,$$

$$\Delta_h\Psi = (\Psi(x+h,y) + \Psi(x-h,y) + \Psi(x,y+h) + \Psi(x,y-h) - \Psi(x,y))/h^2 .$$

The boundary conditions are implemented in an obvious way: $\Psi = 0$ at gridpoints on the boundary, and the normal derivatives at the boundary are approximated by either $\pm D_{0x}\Psi$ or $\pm D_{0y}\Psi$ as appropriate. Thus, e.g.,

$$\omega(x,0) \doteq \Delta_h\Psi(x,0) \doteq 2\Psi(x,h)/h^2 .$$

The difference equations were solved over a range of values of the Reynolds number $R \equiv 1/\varepsilon$, using several different grids. Results for 20 × 20, 30 × 30, and 40 × 40 grids are given in figure 10, which shows $\|\Psi\|_\infty \equiv \max_{i,j} |\Psi_{ij}|$. The curve labeled "best known results" was obtained with very fine grids, at least 100 × 100. The values shown agree very well with those of other calculations using fine grids [11], [12]. Details will appear later [10].

Several facts stand out. All grids produce inaccurate results; the courser grids are less accurate at a given Reynolds number than the finer. On the 20 × 20 grid there is, apparently, only one solution for all values of R shown, but on the other two grids there are values of R (an interval) for which three solutions to the difference equations were found. The curves were traced by a continuation procedure of Keller which has no trouble at limit points, where the solution path turns around (as it does on the two finer grids.)

Motivated by the results for one model problem, we investigate the difference scheme

$$\begin{aligned}\varepsilon\Delta_h\omega = \tfrac{1}{2}\{&D_{0y}\Psi(x+\tfrac{1}{2}h,y)D_{+x}\omega + D_{0y}\,\Psi(x-\tfrac{1}{2}h,y)D_{-x}\omega \\ &- D_{0x}\,\Psi(x,y+\tfrac{1}{2}h)D_{+y}\omega - D_{0x}\,\Psi(x,y-\tfrac{1}{2}h)D_{-y}\omega\}\end{aligned}$$

where $\Psi(x+\tfrac{1}{2}h,y) \equiv \tfrac{1}{2}(\Psi(x,y) + \Psi(x+h,y)$ and $\Psi(x-\tfrac{1}{2}h,y)$, $\Psi(x,y+\tfrac{1}{2}h)$, and $\Psi(x,y-\tfrac{1}{2}h)$ are defined similarly. Thus, for example,

$$\begin{aligned}u(x+\tfrac{1}{2}h,y) &= \Psi_y(x+\tfrac{1}{2}h,y) \\ &\approx \frac{1}{4h}\left(\Psi(x,y+h) - \Psi(x,y-h) + \Psi(x+h,y+h) - \Psi(x+h,y-h)\right) .\end{aligned}$$

Note that this scheme has the same stencil as the previous scheme.

These difference equations were solved on 20 × 20 and 30 × 30 grids. The results shown in figure 11 are substantially better than those for the previous scheme. Not only are the solutions more accurate, there are no spurious multiple solutions on the 30 × 30 grid. This scheme appears to be significantly better, at least for computations of modest accuracy on fairly course grids. A comparison of results for fine grids is planned. Compare also the results for Reynolds number 40 shown in Table 1.

old 121

s&me	grid	$\|\Psi\|_1$	error in $\|\Psi\|_\infty$
old	40	.09982	$.78 \times 10^{-4}$
new	40	.10003	$.57 \times 10^{-4}$
old	121	.10060	———

Table 1. Errors at Re = 40.

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FIGURE 1 L-norm of error, Scheme 1

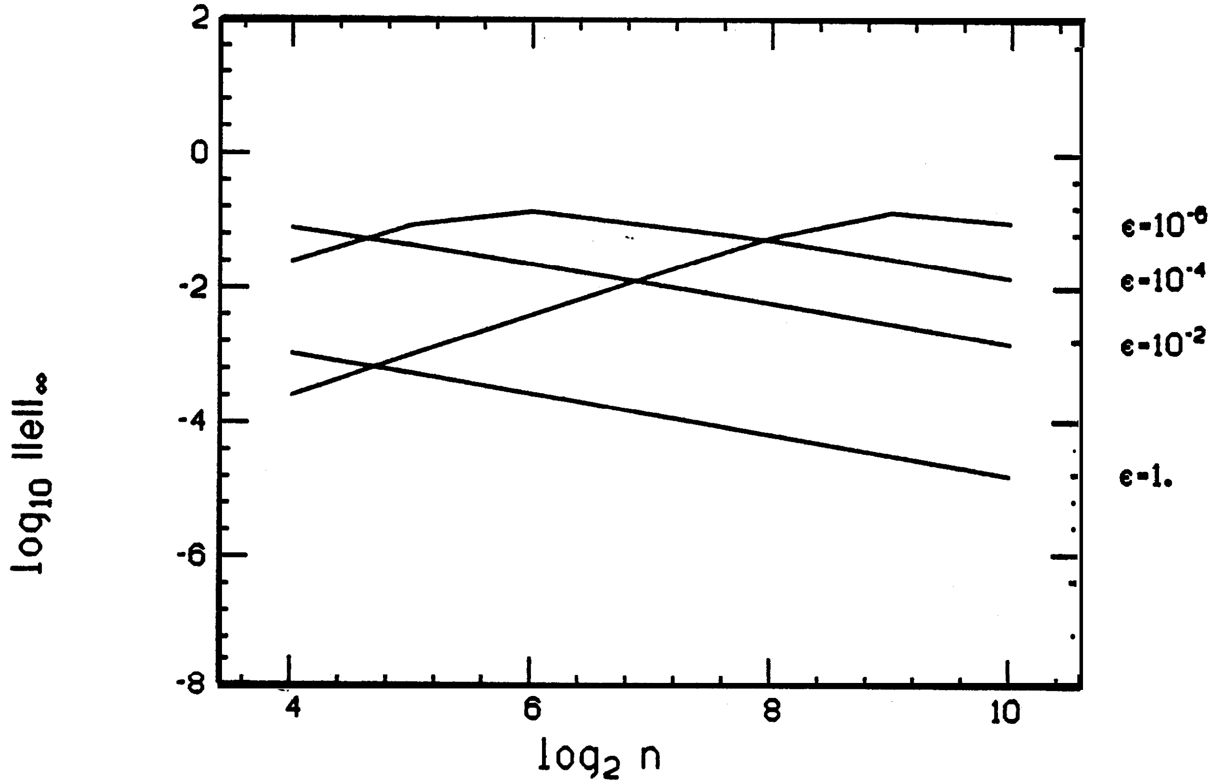


FIGURE 2 L-norm of error, Scheme 2

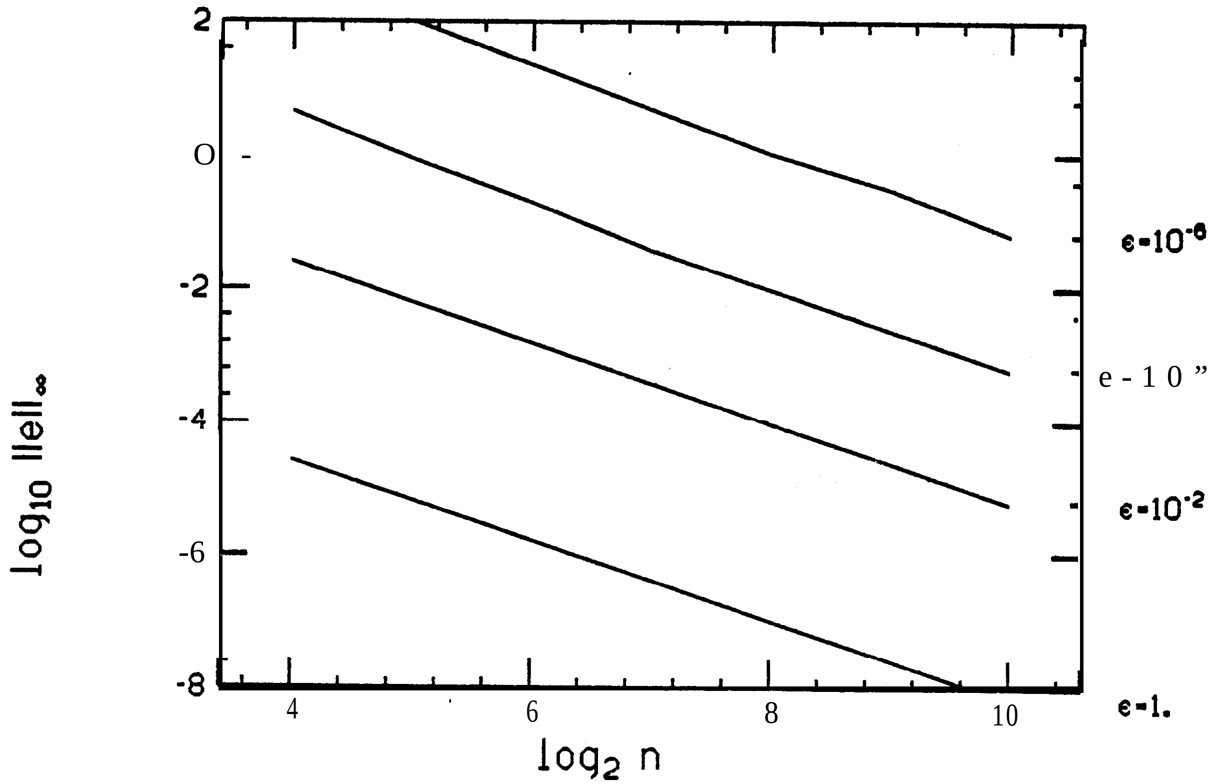


FIGURE 3 L∞-norm of error, Scheme 3

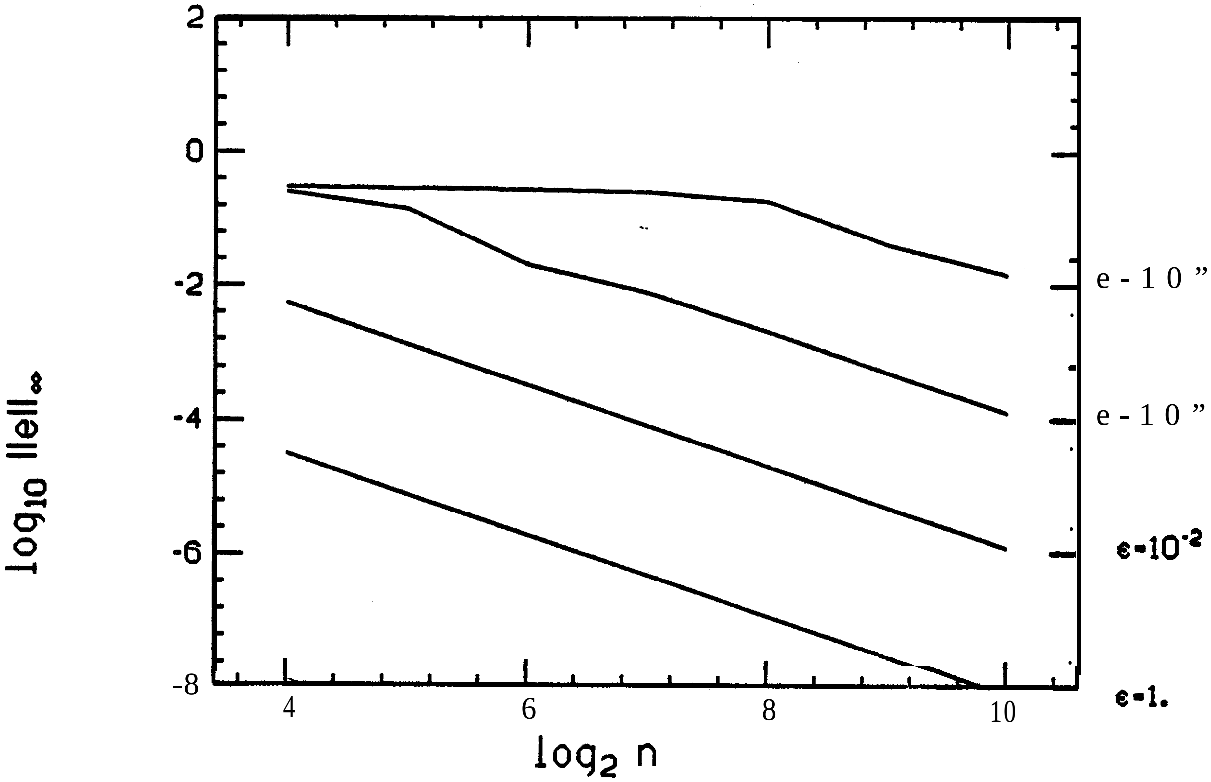


FIGURE 4 L-norm of error, Scheme 4

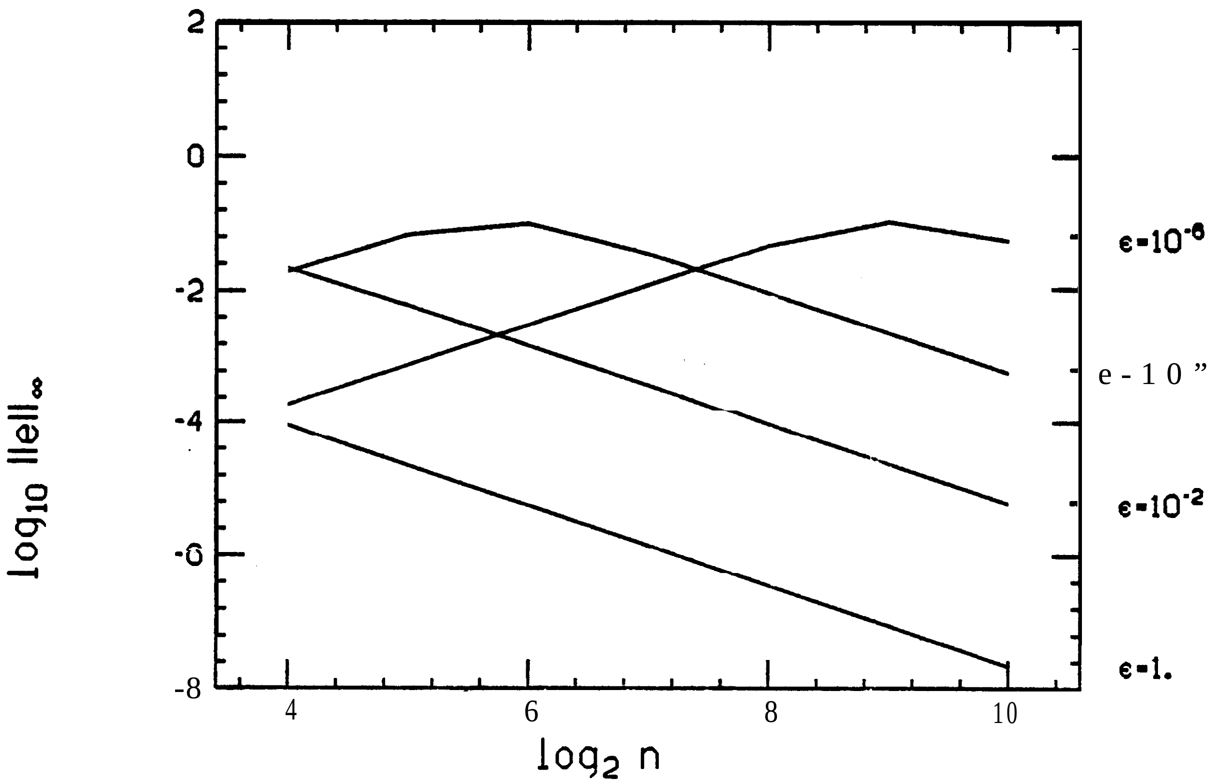


FIGURE 5 L^2-norm of error, Scheme 1

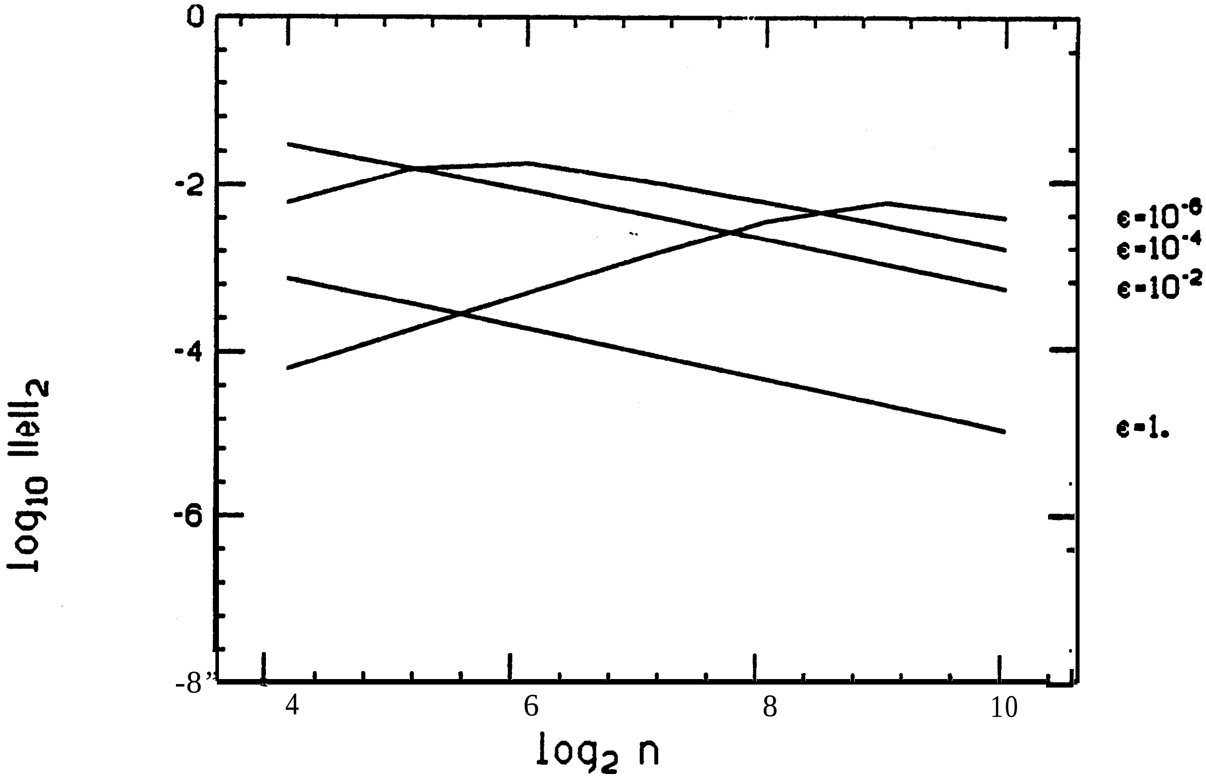


FIGURE 6 L^2-norm of error, Scheme 2

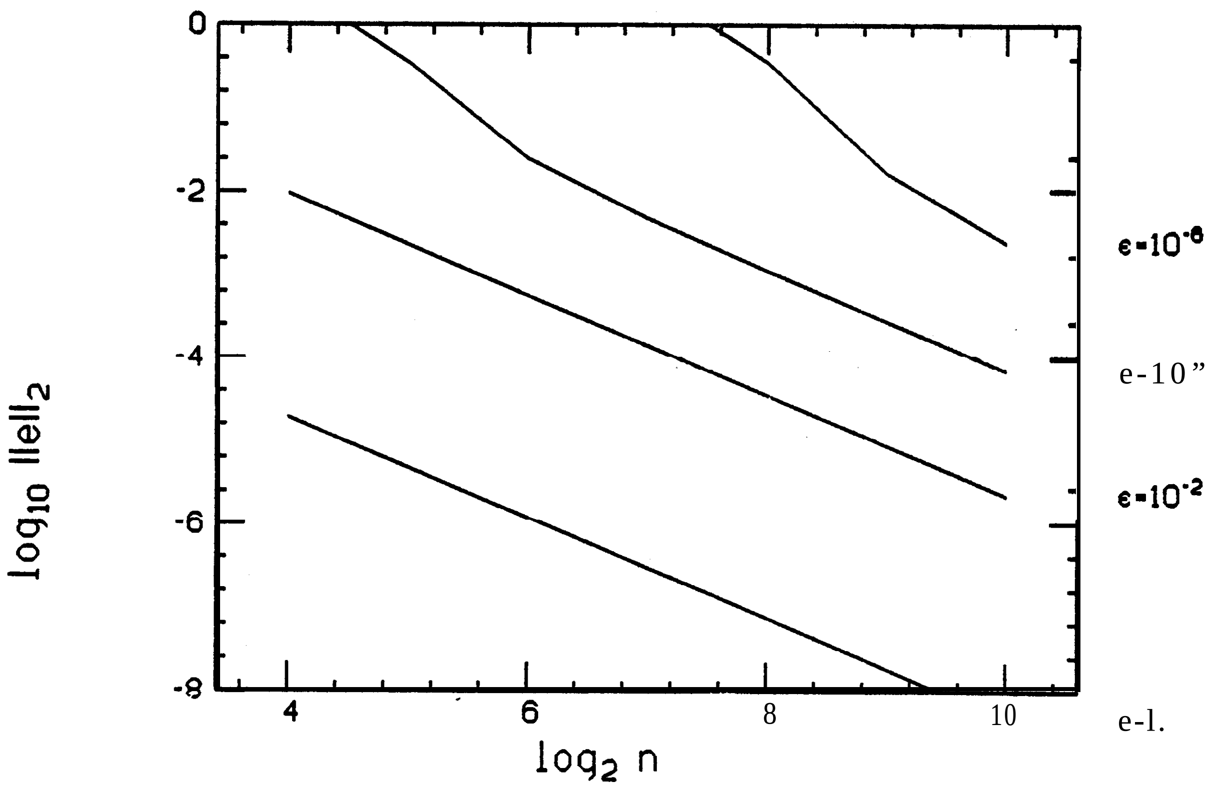


FIGURE 7 L^2-norm of error, Scheme 3

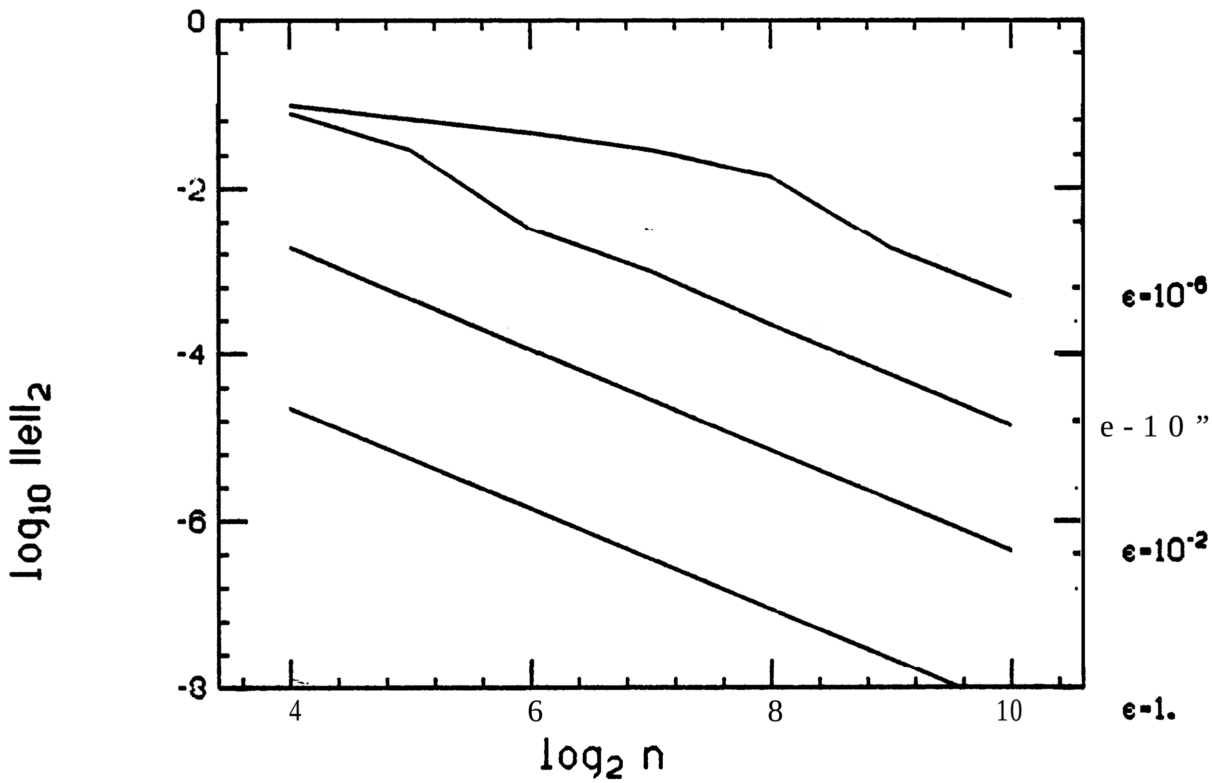
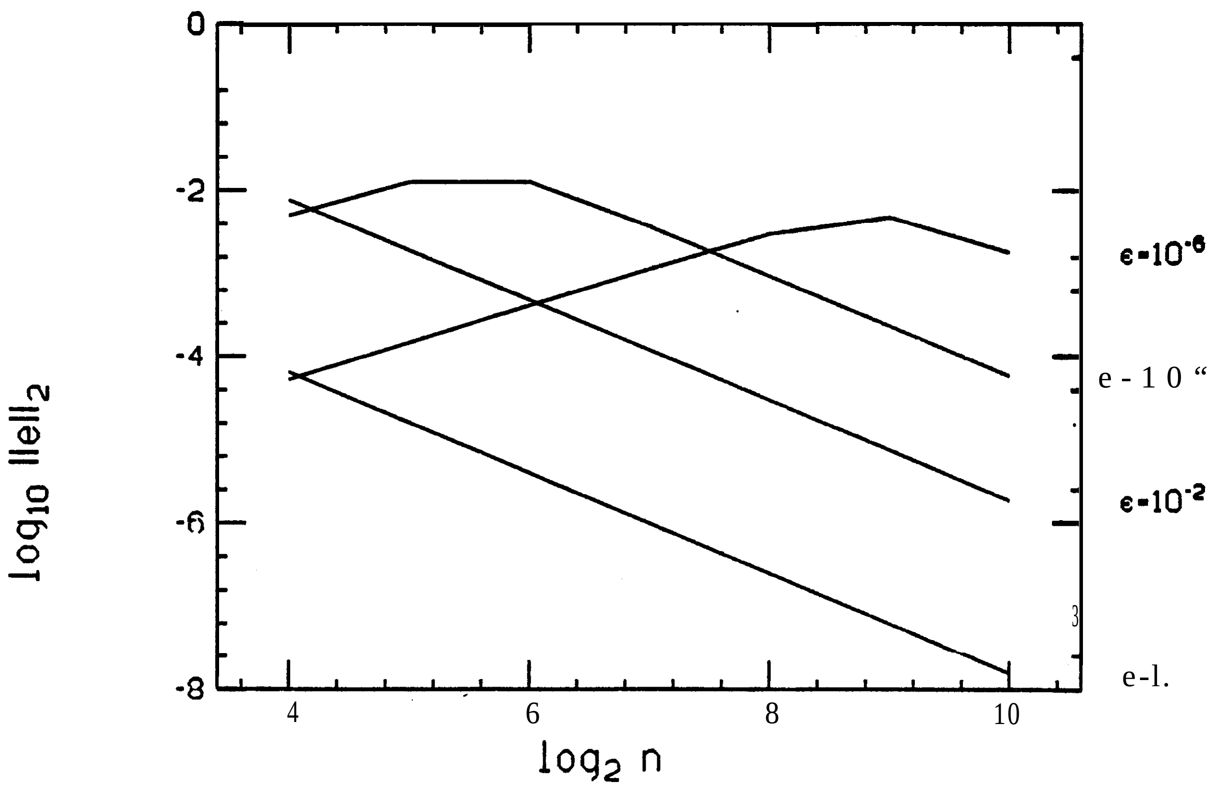


FIGURE 8 L^2-norm of error, Scheme 4



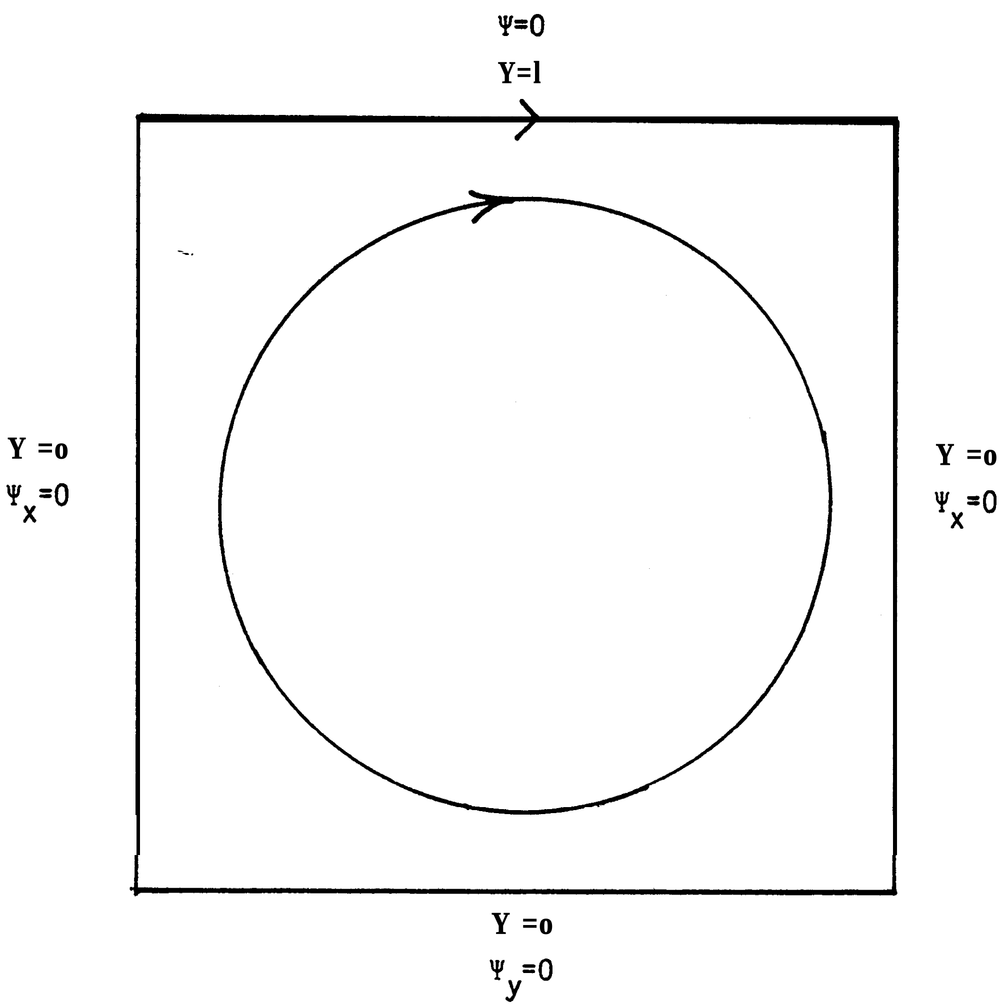


Fig. 9. Driven cavity: boundary conditions.

FIGURE 10

Driven cavity solutions with old scheme

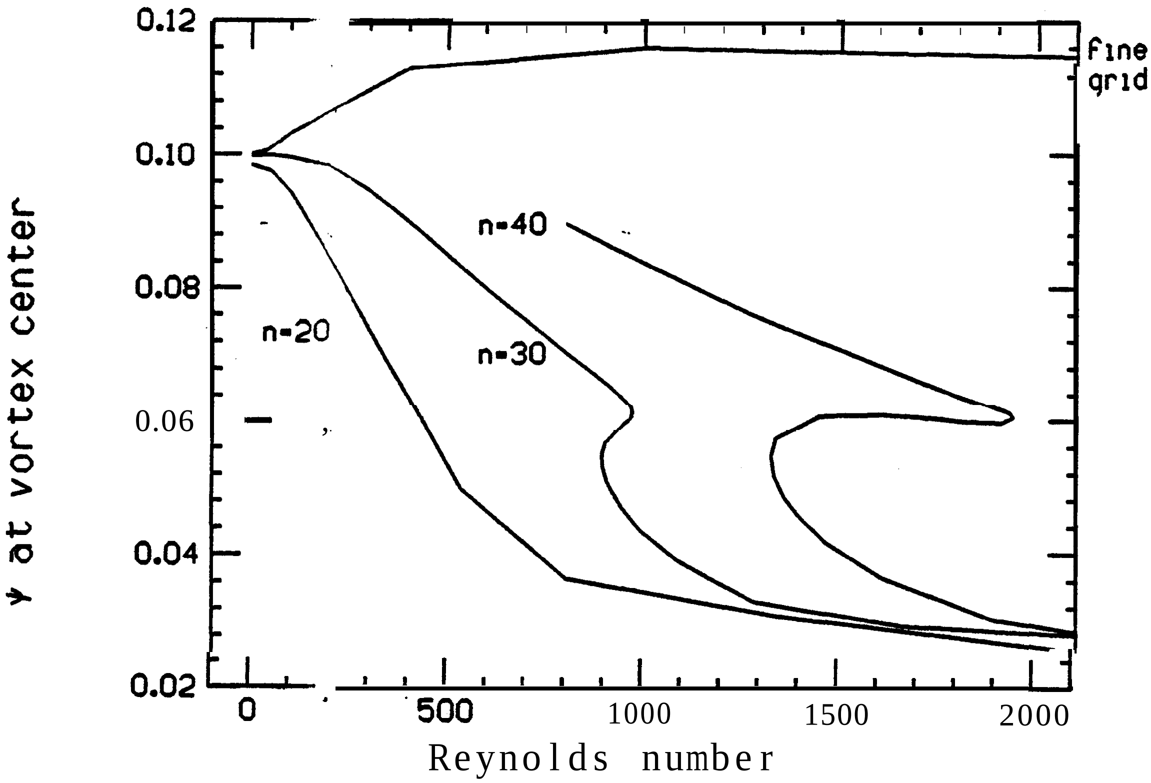


FIGURE 11

Driven cavity solutions with new scheme