On the number of equilibrium placements of mass distributions in elliptic potential fields

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Abstract

Recent papers have demonstrated the use of force fields for mechanical part orientation. The force field is realized on a plane on which the part is placed. The forces exerted on the part's contact surface translate and rotate the part to an equilibrium orientation. Part manipulation by force fields is very attractive since it requires no sensing. We describe force fields that result from elliptic potentials and induce only 2 stable equilibrium orientations for most parts. The proposed fields represent a considerable improvement over previously developed force fields which produced $O(n)$ equilibria for polygonal parts with n vertices. The successful realization of these force fields could significantly affect part manipulation in industrial automation.

1 Introduction

Part orientation is an important and time-consuming operation in manufacturing. Parts and, in particular, small parts arrive at the assembly site in boxes, and they have to be sorted before they can be used in the manufacturing cycle. Traditionally part orientation has been performed with Vibratory Part Feeders [7]. These are mechanical devices designed for the orientation of a single part or of a small number of parts. They rely on a vibratory motion to force parts inside tracks that have built-in mechanical filters. These filters allow only the parts with the right orientation to go through them.

The automation of the sorting process has recently attracted a lot of attention. Goldberg [5] showed how to orient a part with the use of parallel jaw grippers. The approach requires no sensing since the gripper actions have no feedback. In [5] it is proved that the shortest sequence of gripper actions that orient a part with $O(n)$ vertices can be found with an $O(n^2 \log n)$ algorithm. The orientation is done up to symmetries of the convex hull of the part. The length of the orientation sequence is $O(n^2)$.

Böhringer et al. [1, 2, 3, 4] have investigated the use of force-fields for part orientation. The force field is realized on a plane and the part is placed on this plane with one of its surfaces touching the plane. The force exerted on that surface makes the part translate and rotate on the plane. Part orientation by force fields is very attractive since it requires no sensing. In [4] Microfabricated Actuator Arrays (MEMS) are used to implement force fields for part orientation.

In that paper, the authors developed the concept of "squeeze" fields. These fields can orient parts in a way similar to the way that a parallel jaw gripper orients parts. In [2, 3] "squeeze" and "radial" fields, as well as general manipulation strategies for part orientation are investigated. In the same papers, it is demonstrated that microelectromechanical structures are capable of implementing some of these force fields. Liu et al. [6] have also worked with the concept of a dense array of programmable mechanisms. They call their array an "intelligent motion surface". They explore the capabilities this surface and MEMS technologies for implementing it. It is worth noting that force fields may also be realized through different technologies. For example, [1] investigates the use of vibratory plates to create certain force fields.

This paper shows that elliptic potentials can generate force fields that induce a constant number of equilibrium placements for most parts. In particular, it is shown that "asymmetric" parts (see discussion in section 2.4) have only 2 stable equilibria in the proposed force fields. This result represents a considerable improvement over previously considered force fields which induced $O(n)$ equilibria for polygonal parts with n vertices [3]. Furthermore, the analysis provides a way to take into account properties of the part (for example, varying friction coefficients over its contact surface or its weight), provided that a good model for these is available. The equilibrium placements of a part can be computed a priori with numerical methods or, in certain cases, analytic calculations. Our work makes no assumptions about the shape of the part to be manipulated or its connectivity. The only assumption made is that the part has a surface in contact with the force field and also that the part is rigid: even if it consists of multiple bodies, these bodies are considered rigidly attached to each other. The realization of the proposed force fields is challenging. At a microscopic level, it may be possible to implement these fields in the future with MEMS technology [2, 3]. At a macroscopic level, one can imagine the implementation of these fields with an array of motors (see also section 3). Borrowing our terminology from physics, we refer below to parts as "mass" distributions over $\mathbb{R}^2$. The discussion in section 2 is general. Its implications for part orientation are discussed in section 3.

2 Equilibrium Placements in Elliptic Potential Fields

Let $w : \mathbb{R}^2 \to \mathbb{R}$ be a "mass" distribution function. For our analysis we require that $w(x, y) \geq 0$, for $x, y \in \mathbb{R}$, and $W = \int_{\mathbb{R}^2} w(x, y) dxdy < \infty$. Let us also call

$$\mathbf{c} = (c_x, c_y)^\top = \frac{1}{W} \int_{\mathbb{R}^2} (x, y)^\top \, w(x, y) \, dx \, dy \tag{1}$$

the "center of mass" of the distribution w, and define the following quantities if w:

$$s_{mn} = s_{mn}(w) = \int_{\mathbb{R}^2} x^m y^n w(x, y) \, dx \, dy. \tag{2}$$

Only s_{11}, s_{20} and s_{02} will be relevant in our discussion and we assume that they are finite.

We shall investigate the conditions for equilibrium for the mass distribution w in the presence of a force field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$. It is assumed that the force field $\mathbf{F}$ is realized in a plane in such a way that the force exerted on a domain $\Omega \subseteq \mathbb{R}^2$ is $\int_\Omega \mathbf{F}(x, y) w(x, y) \, dx \, dy$.

We are interested in knowing which placements of a given mass distribution in a fixed force field (of a very specific type) give rise to equilibrium. For convenience we assume that the distribution is given at an initial position where $\mathbf{c} = \mathbf{0}$ and, from now on, we use the notation $\mathbf{r} = (x, y)^\top$. By "a placement of $w(\mathbf{r})$" we mean a mass distribution of the type $w(A_\theta \mathbf{r} + \mathbf{t})$, where the matrix A_θ is rotation by $\theta \in [0, 2\pi)$ and $\mathbf{t} \in \mathbb{R}^2$ is a translation vector. The force field $\mathbf{F}$ under consideration will be chosen so that the number of equilibria of w in $\mathbf{F}$, i.e. the number of parameters $(\theta, \mathbf{t})$, is as small as possible.

2.1 Equilibrium conditions

For a mass distribution to be in equilibrium we require that (**a**) the total force exerted on the distribution is zero and (**b**) the total moment about, say, the origin is zero. That is, we require that the following two equations hold:

$$\int \mathbf{F}(\mathbf{r}) w(\mathbf{r})\, dx\, dy = \mathbf{0}, \tag{3}$$

and

$$\int \mathbf{F}(\mathbf{r}) \times \mathbf{r}\, w(\mathbf{r})\, dx\, dy = \mathbf{0}, \tag{4}$$

where from now on all integrals extend over $\mathbb{R}^2$.

2.2 Elliptic potential fields

With hindsight we consider a force field of the type

$$\mathbf{F}(x, y) = (-\alpha x, -\beta y), \tag{5}$$

where α and β are two <u>distinct</u> positive constants. Figure 1(a) displays one such force field with $\alpha = 1$ and $\beta = 2$. The magnitude of the force exerted at (x, y) is also plotted in Figure 1(b). Note that this vector field is the negative gradient of the elliptic potential

$$f(x, y) = \frac{\alpha}{2} x^2 + \frac{\beta}{2} y^2.$$

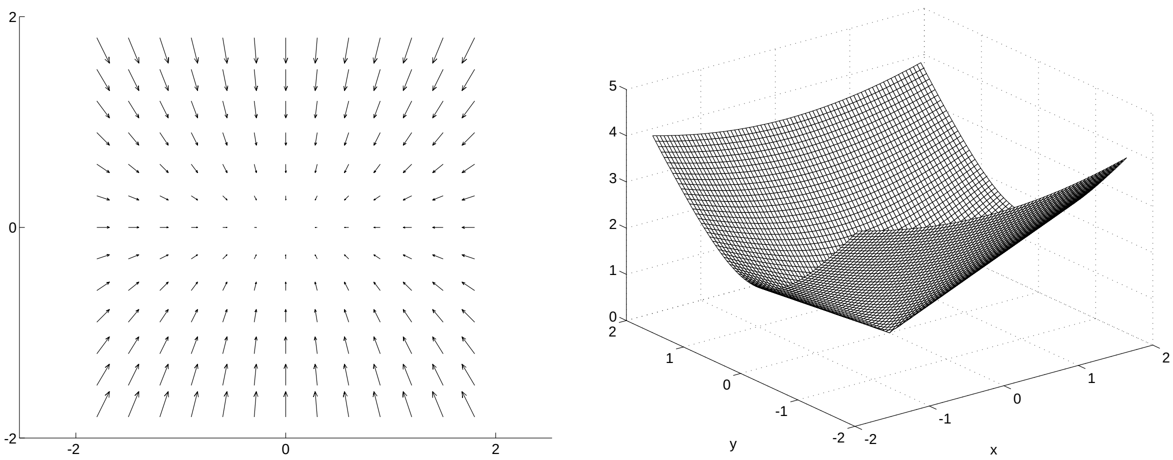
This potential is plotted in Figure 2, for $\alpha = 1$ and $\beta = 2$. The use of potential fields for manipulation tasks was proposed in [2, 3].

2.3 Force equilibrium

If $c_x = \frac{1}{W} \int x w(x, y)\, dx\, dy$ and $c_y = \frac{1}{W} \int y w(x, y)\, dx\, dy$ are the coordinates of the center of mass of w (see (1) above), the total force exerted on w, given by the left hand side of (3), is equal to

$$(-\alpha W c_x, -\beta W c_y).$$

Condition (3) is thus equivalent to the center of mass of the distribution w being equal to $\mathbf{0}$. But the center of mass of the general distribution $w(A_\theta \mathbf{r} + \mathbf{t})$ in our class is clearly $\mathbf{t}$, therefore, in looking for equilibrium placements of w, one only needs to consider the placements of the type $w(A_\theta \mathbf{r})$.



(a) (b)

Figure 1: (a) Force field for $\alpha = 1$ and $\beta = 2$, (b) Magnitude of the force of the same field.

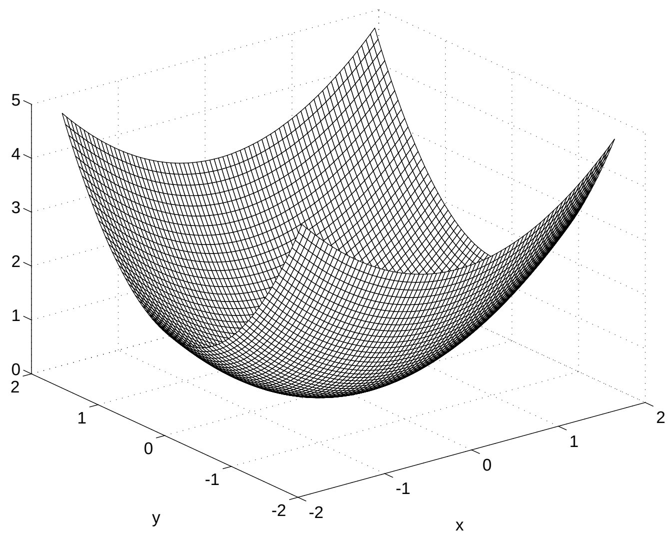


Figure 2: Elliptic potential for $\alpha = 1$ and $\beta = 2$.

2.4 Force and moment equilibrium

Having established that all distributions of the type $w(A_\theta \mathbf{r})$ satisfy condition (3) we now pass to the investigation of condition (4). It will turn out that, for "most" mass distributions w and for whatever distinct positive values of α and β, there are exactly 4 values of θ for which (4) holds.

The matrix A_θ is equal to $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Making the change of variable $(u,v)^\top = A_{-\theta}(x,y)^\top$ and renaming the variables u, v again as x, y, the total moment of the mass distribution $w(A_\theta \mathbf{r})$ becomes

$$\begin{aligned}
\mathbf{M} &= \int \mathbf{F}(\mathbf{r}) \times \mathbf{r}\, w(A_\theta \mathbf{r})\, dx\, dy \\
&= \int \mathbf{F}(A_{-\theta}\mathbf{r}) \times (A_{-\theta}\mathbf{r})\, w(\mathbf{r})\, dx\, dy \\
&= \int \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\alpha(x\cos\theta + y\sin\theta) & -\beta(-x\sin\theta + y\cos\theta) & 0 \\ x\cos\theta + y\sin\theta & -x\sin\theta + y\cos\theta & 0 \end{vmatrix} w(x,y)\, dx\, dy \\
&= \int (\beta-\alpha)(x\cos\theta + y\sin\theta)(-x\sin\theta + y\cos\theta) w(x,y)\, dx\, dy \cdot \mathbf{k} \\
&= \int (\beta-\alpha)(-x^2\cos\theta\sin\theta + xy\cos^2\theta - xy\sin^2\theta + y^2\cos\theta\sin\theta)\, w(x,y)\, dx\, dy \cdot \mathbf{k} \\
&= (\beta-\alpha)\int \left((y^2-x^2)\frac{\sin 2\theta}{2} + xy\cos 2\theta\right)\, w(x,y)\, dx\, dy \cdot \mathbf{k} \\
&= (\beta-\alpha)\left(\frac{\sin 2\theta}{2}\int (y^2-x^2)\, w(x,y)\, dx\, dy + \cos 2\theta \int xy\, w(x,y)\, dx\, dy\right) \cdot \mathbf{k}.
\end{aligned}$$

Thus, since $\alpha \neq \beta$, we have $\mathbf{M} = \mathbf{0}$ if and only if

$$\frac{s_{02} - s_{20}}{2}\sin 2\theta + s_{11}\cos 2\theta = 0. \tag{6}$$

Equivalently, we want the vectors $(\cos 2\theta, \sin 2\theta)$ and $(s_{11}, \frac{1}{2}(s_{02} - s_{20}))$ to be orthogonal. We now have to distinguish two cases.

Symmetry: $s_{11} = 0$ and $s_{02} = s_{20}$.
Clearly in this case (6) is satisfied for all $\theta \in [0, 2\pi)$ and we have equilibrium regardless of orientation.

Asymmetry: $s_{11} \neq 0$ or $s_{02} \neq s_{20}$.
When θ goes from 0 to 2π the vector $(\cos 2\theta, \sin 2\theta)$ traverses the unit circle twice. The two vectors, $(\cos 2\theta, \sin 2\theta)$ and $(s_{11}, \frac{1}{2}(s_{02} - s_{20}))$ will be orthogonal for exactly 4 values of θ, say $\theta_1 = \theta_0$, $\theta_2 = \theta_0 + \pi$, $\theta_3 = \theta_0 + \frac{\pi}{2}$, and $\theta_4 = \theta_0 + \frac{3\pi}{2}$. In addition, either the first pair of them is stable and the second unstable, or vice versa. The reason is that the sign of the left hand side of (6) determines the direction in which moment $\mathbf{M}$ rotates the mass distribution. If this sign is positive, $\mathbf{M}$ rotates the mass distribution counter-clockwise, else the rotation is done clockwise.

While $(\cos 2\theta, \sin 2\theta)$ is rotated around the vector $(s_{11}, \frac{1}{2}(s_{02} - s_{20}))$ the sign of the left hand side of (6) changes after the two vectors attain an orthogonal orientation. Hence, we observe sign changes of the left hand side of (6) for the 4 values of θ given above. Let θ_1 and θ_2 be the roots of (6) for which the sign of its left hand side changes from a positive value to a negative value while moving in a counter-clockwise direction. Then these values indicate stable equilibrium placements of the mass distribution, since $\mathbf{M}$ will force the mass at the same equilibrium after a small rotational perturbation. In this case, θ_3 and θ_4 are unstable placements since after a small perturbation around them, $\mathbf{M}$ will rotate the mass away from θ_3 or θ_4, to one of θ_1 or θ_2.

In summary we have proved the following.

Theorem 1 *Let $w : \mathbb{R}^2 \to \mathbb{R}$ be a nonnegative "mass" distribution with finite s_{ij} with $i + j \leq 2$ and whose "center of mass" is at $\mathbf{0}$, and let $\mathbf{F}(x, y) = (-\alpha x, -\beta y)$, with $\alpha \neq \beta$, $\alpha > 0$, $\beta > 0$, be the underlying force field.*

SYMMETRY*: If $s_{11} = s_{20} - s_{02} = 0$ the "mass" distribution $w(A_\theta \mathbf{r} + \mathbf{t})$ is at (force and moment) equilibrium whenever $\mathbf{t} = \mathbf{0}$.*

ASYMMETRY*: Otherwise, the distribution $w(A_\theta \mathbf{r} + \mathbf{t})$ is in equilibrium only when $\mathbf{t} = 0$ and for exactly 4 distinct values of $\theta \in [0, 2\pi)$. These 4 values of θ are $\frac{\pi}{2}$ apart and only 2 of them, say θ_0 and $\theta_0 + \pi$, represent stable equilibria, the others, $\theta_0 + \frac{\pi}{2}$ and $\theta_0 + \frac{3\pi}{2}$ being unstable.*

3 Part Orientation

In practice, we seek to orient a part of finite shape with the use of the force fields described in the previous section. If $w(x, y)$ is the support function of the part, then all the requirements of Theorem 1 are satisfied. It is also very easy to compute with numerical techniques the values of s_{11}, s_{20}, and s_{02} and predict, for a given part, whether it will have 2 stable equilibria in the force field considered. The equilibrium orientations can also be calculated. Note that the equilibrium placements of a part are independent of α and β.

In many cases it is clear that a part will have many equilibrium orientations. For example, consider a part whose contact surface with a force field is a regular n-gon. This part will be at equilibrium when its "center of mass", as defined above, is at $\mathbf{0}$ no matter what its orientation is. The "center of mass" in this case is the center of its n-gon contact surface. Suppose now that the part had only two equilibria θ_0 and $\theta_0 + \pi$ and that the part is at equilibrium θ_0. If we rotate the part by $\frac{2\pi}{n}$ then we should have an equilibrium again, due to the symmetry of the part. Hence, since this part can not have only two equilibrium orientation it must be in equilibrium for any value of θ, according to Theorem 1. Indeed, for this part, it can be shown that $s_{11} = s_{20} - s_{02} = 0$.

Importantly, our analysis provides a way to take into account properties of the part. If, for example, the friction coefficient is varying over the contact surface of the part, then w can be used to encode this information. Or, if there is a simple relation between the weight of the part above (x, y) and the force exerted at (x, y), then w can be used to represent this relation. In general, provided that a good model is available, the use of w can be of great practical importance.

The realization of these force fields presents us with a challenge. At a macroscopic level, it may be possible to implement these fields with a $n \times n$ array of motors, each of which has the orientation of the force it should exert. The magnitude of the force needs to be controlled individually. At a microscopic level, it may be possible to implement such an array in the future with MEMS technology [4, 2, 3]. If the magnitude and orientation of the force exerted by every pixel in this array can be controlled, the realization of these fields will be easy: every pixel will instructed to exert a force with F_x coordinate equal to α multiplied by the x coordinate of the pixel exerting the force, and F_y coordinate equal to β multiplied by the y coordinate of the pixel. If it is only possible to specify a force in one of the x or y directions at each pixel, then two arrays, one controlled only in the x direction and the other controlled only in the y direction can be "interleaved". If the arrays are dense, the resulting force will be a force with the desired magnitude and direction. Once a single array is constructed, it can be used for orienting many parts. In fact, since the final orientation of a part can be predicted beforehand, the orientation of the array can be changed so that the part will end up in the desired orientation for the assembly task.

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