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Verification of Concurrent Programs, Part I: The Temporal Framework

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VERIFICATION OF CONCURRENT PROGRAMS: THE TEMPORAL FRAMEWORK

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ABSTRACT

This is the first in a series of reports describing the application of temporal logic to the specification and verification of concurrent programs.

We first introduce temporal logic as a tool for reasoning about sequences of states. Models of concurrent programs based both on transition graphs and on linear-text representations are presented and the notions of concurrent and fair executions are defined.

The general temporal language is then specialized to reason about those execution sequences that are fair computations of a concurrent program. Subsequently, the language is used to describe properties of concurrent programs.

The set of interesting properties is classified into invariance (safety), *eventuality* (liveness), and *precedence* (until) proper ties. Among the properties studied are: partial correctness, global invariance, clean behavior, mutual exclusion, absence of deadlock, termination, total correctness, intermittent assertions, accessibility, responsiveness, safe liveness, absence of unsolicited response, fair responsiveness, and precedence.

In the following reports of this series, we will use the temporal formalism to develop proof methodologies for proving the properties discussed here.

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INTRODUCTION

Temporal logic is a special branch of logic that deals with the development of situations in time. Whereas ordinary logic is adequate for describing a *static* situation, temporal logic enables us to discuss how a situation *changes* due to the passage of time. An execution of a program is precisely a chain of situations, called execution states, that undergo a series of transformations determined by the program's instructions. This suggests that temporal logic is an appropriate tool for reasoning about the execution of programs. The special advantage of this approach is that it enables us to formalize the entire *execution* of a program and not just the *function* or *relation* it computes.

The temporal logic approach offers special advantages for the formalization and analysis of the behavior of *concurrent programs.* Concurrent programs have long been a difficult subject to formalize and have often defied generalization of methods that worked perfectly for sequential programs.

One inherent difficulty in analyzing a concurrent program is that when combining two processes to be run in parallel, we cannot infer the *input-output relation* computed by the combined program from just the input-output relations computed by each of the individual component processes. The obvious reason for this is that, running in parallel, the processes may interfere with one another, altering the behavior each would have when run alone. Consequently, in order for any approach to stand a chance of success, it must deal with more than the input-output relation computed by a program. It should be concerned with *execution sequences* in one form or another, as well as be able to discuss mid-execution events.

Another inherent difficulty is the *discontinuity* associated with the simulation of concurrency by *multiprogramming.* A very convenient and widely used model of real concurrency is to regard the participating events as composed of many atomic basic steps. Then instead of requiring that these basic steps occur concurrently, we consider sequences in which these steps are *interleaved* in all possible ways. The problem with modelling concurrency by multiprogramming (interleaving) is that without further restrictions a certain process can be discriminated against by having its execution continually delayed. Disallowing this discrimination introduces a discontinuity into the set of interleaved execution sequences.

Consequently, any approach which is based strongly on the concept of continuity, such as the denotational approach or equivalent relational ones, is bound to face severe difficulties when extended to deal with concurrency.

Temporal logic avoids both these difficulties by (a) being geared from the start to analyze and formalize properties in terms of execution sequences, and (b) not being based on limits and assurnptions of continuity. In fact, it can very easily and naturally express such concepts as "eventually" which describes an event arbitrarily ahead in the future, but still a finite duration away.

In this report we introduce the framework and language of temporal logic and demonstrate its appropriateness for describing properties of programs.

We start with an exposition of *modal logic* whose domain of interpretation is a set of *states* and (general) *accessibility relations* connecting these states. We then specialize to *temporal logic* which requires that the states form a *linear discrete sequence.* Linear discrete sequences can be used to describe a dynamic process that goes through changes at discrete instants. Consequently, temporal logic is suitable for reasoning about such dynamic processes and their behavior in time.

Next, we present a model *of concurrent programs.* The basic model is based on several concurrent processes, each of which is given in the form of a transition graph or a linear-text program. Executions of concurrent programs are defined to be an *interleaving* of execution steps, each taken from one of the processes. We discuss the conditions under which an interleaved execution faithfully represents real concurrency. One of these conditions calls for the interleaving to be *fair* in that no process is neglected for too long.

We then show how the language of temporal logic can be further specialized to reason about *execution sequences* of programs. In this way, properties of programs which are expressible as properties of their execution sequences are readily formalizable.

The rest of the report overviews in a systematic manner the different properties of interest. They are classified into:

- *Invariance properties,* stating that some condition holds continuously throughout the computation.

- *Eventuality properties,* stating that under some initial conditions, a certain event (such as the program's termination) must eventually be realized.

- *Precedence properties,* stating that a certain event always precedes another.

For each class of properties, we present several typical and useful properties together with sample programs illustrating these properties.

1. THE GENERAL CONCEPTS OF TEMPORAL LOGIC

In the development of logic as a formalization tool, we can observe an increasing ability to express change and variability. *Propositional Calculus* was developed to express constant or absolute truth, stating basic facts about the universe of discourse. The propositional framework mainly deals with the question of how the truth of a composite sentence depends on the truth of its constituents. In *Predicate Calculus* we deal with variable or relative truth by distinguishing the statement (the predicate) from its arguments. It is understood that the statement may be true or false according to the particular individuals it is applied to. Thus we may regard predicates as parameterized propositions. The *Modal Calculus* adds another dimension of variability to this description by predicates. If we contemplate a major transition in which not only individuals, but also the meaning of functions and predicates are changed, then the modal calculus provides a special notation for this major change. For instance, any chain of reasoning which is valid on Earth may become invalid on Mars because some of the basic concepts naturally used on Earth may assume completely different meanings (or become meaningless) on Mars. Conceptually, this calls for a partition of the universe of discourse into worlds of similar structure but different contents. Variability within a world is handled by changing the arguments of predicates, while changes between worlds are expressed by the special modal formalism.

Consider for example the statement: "It rains today". Obviously, the truth of such a statement depends on at least two parameters: The date and the location at which it is stated. Given a specific date t_0 and location ℓ_0, the specific statement: "It rains at ℓ_0 on t_0" has propositional character, *i.e.*, it is fully specified and must either be true or false. We may also consider the fully variable predicate $rain(\ell, t)$: "It rains at ℓ on t" which gives equal priority to both parameters. The modal approach distinguishes two levels of variability. In this example, we may choose time to be the major varying factor, and the universe to consist of worlds which are days. Within each day we consider the predicate $rain(\ell)$ which, given the date, depends only on the location. Alternatively, we can choose the location to be the major parameter and regard the raining history of each location as a distinct world.

As is seen from this example, the transition from predicate logic to modal logic is not as sharp as the transition from propositional logic to predicate logic. For one thing it is not absolutely essential. We could manage quite reasonably with our two parameter predicate. Second, the decision as to which parameter is chosen to be the major one may seem arbitrary. It is strongly influenced by our intuitive view of the situation.

In spite of these reservations there are some obvious advantages to the introduction and use of modal formalisms. It allows us to explicitly make one parameter more significant than all the others, and makes the dependence on that parameter implicit. Nowadays, when increasing attention is being paid to the clear correspondence between the syntactical structure of a program and its functional decomposition (as is repeatedly stressed by the discipline of structured programming), it seems only appropriate to introduce extra structure into the description of varying situations. Thus a clear distinction is made between variation within a world, which we express using predicates and quantifiers, and variation from one world to another, which we express using the modal operators.

Another way to view the generalization offered by modal logic is to claim that predicate calculus is appropriate for describing *static situations*. It gives statements about basic objects and their interrelation. The additional dimension provided by the modal logic is that of *dynamic change* from one situation into the other. One of the characteristics of changes due to time transitions is the fact that the same basic objects and entities exist in each of the static situations but that their attributes and interrelations may change. Thus modal logic faithfully and conveniently portrays for us a *dynamic situation* consisting of a set of static situations and rules of change between them.

THE MODAL FRAMEWORK

The general modal framework ([HC]) considers a *universe* that consists of many similar *states* (or *worlds*) and a basic *accessibility relation* between the states, $R(s, s')$, which specifies the possibility of getting from one state s to another state s'.

Consider again the example of rainy days, with time taken to be the major parameter. There, each state in the universe is a day. A possible accessibility relation might hold between two days s and s' if s' is in the future of s.

The main notational idea is to avoid any explicit mention of either the state parameter (date in our example) or the accessibility relation. Instead we introduce two special operators that describe properties of states which are accessible from a given state in a universe.

The two *modal operators* introduced are $\Box$ (called the *necessity operator*) and $\Diamond$ (called the *possibility operator*). Their meaning is given by the following rules of interpretation in which we denote by $|w|_s$ the truth value of the formula w in a state s:

$$|\Box\, w|_s = \forall s'[R(s, s') \supset |w|_{s'}]$$

$$|\Diamond\, w|_s = \exists s'[R(s, s') \wedge |w|_{s'}]\ .$$

Thus, $\Box\, w$ is true at a state s if the formula w is true at all states R-accessible from s. Similarly, $\Diamond\, w$ is true at a state s if w is true in at least one state R-accessible from s. Usually, R is taken to be reflexive, so that every state is R-accessible from itself and thus $R(s,\ s)$ always holds.

A *modal formula* is a formula constructed from proposition symbols, predicate symbols (including equality), function symbols, individual constants and individual variables, the classical operators and quantifiers, and the modal operators. A formula without any modal operators is called a *static* formula. A fully modal (*dynamic*) formula is conveniently viewed as consisting of static subformulas to which modal and classical operators are applied. The truth value of a modal formula at some state of a given universe is found by a repeated use of the rules above for the modal operators and evaluation of any static subformula on the state itself. It is assumed that every state contains a full interpretation of all the classical symbols in the formula, which fully determines the truth value of every static formula.

For example, the formula

$$rain(\ell) \supset \Diamond \sim rain(\ell)$$

is interpreted in our model of rainy days as stating: For a given day and a given location ℓ, if it rains on that day at ℓ then there exists another day in the future on which it will not rain at ℓ; thus any rain will eventually stop. Similarly,

$$rain(\ell) \supset \Box\, rain(\ell)$$

claims that if it rains on that day it will rain everafter. Note that any modal formula is always considered with respect to some fixed reference state, which may be chosen arbitrarily. In our example, it has the meaning of "today".

Consider the general formula

$$\Box \sim w \equiv \sim \Diamond w.$$

As we can see from the definitions this claims that all R-accessible states satisfy $\sim w$ if and only if there does not exist an R-accessible state satisfying w. This formula is true in any state for any universe with an arbitrary R.

We now give a more precise definition. A *universe* U for a modal formula w consists of a nonempty domain D, a set of *states* (or *worlds*) S, and a binary relation R on S, called the *accessibility relation.* Each state s provides a first-order interpretation over the domain D for all the proposition symbols, predicate symbols, function symbols, individual constants, and (free) individual variables in w. A *model* (U, s_0) is a universe U with one of the states of U, $s_0 \in S$, designated as the initial or reference state. In short,

$$\text{universe of } w = \begin{cases} \text{domain} - D \\ \text{set of states} - S \\ \text{accessibility relation between states} - R \end{cases}$$

where

state = assignment to symbols of w over D

We define the truth value of a modal formula w at a state s (denoted by $|w|_s$) in a given universe U inductively:

1. If w is static, *i.e.*, contains no modal operators, then its truth value $|w|_s$ is found by interpreting w in s.
2. $|\Box w|_s$ is $\forall s'[R(s, s') \supset |w|_{s'}]$.
3. $|\Diamond w|_s$ is $\exists s'[R(s, s') \wedge |w|_{s'}]$.
4. $|w_1 \vee w_2|_s$ is true *iff* either $|w_1|_s$ is true or $|w_2|_s$ is true.
5. $|\sim w|_s$ is true *iff* $|w|_s$ is false.

Note that by our rules of interpretation

- $|\Diamond(\Box w)|_s$ means that $|\Box w|_{s'}$ is true at some state s', R-accessible from s. That is,

$$\Diamond \Box w$$

stands for: *we can get to a point where w is true everafter*; *i.e.*, there is a state s' R-accessible from s such that s' itself and all of its R-descendants satisfying w.

- $|\Box(\Diamond w)|_s$ means that $|\Diamond w|_{s'}$ is true for all states s', R-accessible from s. That is,

$$\Box \Diamond w$$

stands for: *wherever we go w is still realizable*; *i.e.*, for every state s' accessible from s it is possible to find an R-descendant of s' which satisfies w.

- $|\Box(w \supset \Box w)|_s$ means that $|w \supset \Box w|_{s'}$ is true for all states s', R-accessible from s. That is,

$$\Box(w \supset \Box w)$$

stands for: if w ever becomes true in some s' accessible from s, it remains true for all descendants of s'.

If a formula w is true in a state s_0 in a universe U we say that (U, s_0) is a *(satisfying) model* for that formula, or that the formula is *satisfied* in (U, s_0).

A formula w which is true in all states of every universe is called *valid*; that is, for every universe U of w and for every state s in U, $|w|_s$ is true. For example, the formula

$$\Box \sim w \equiv \sim \Diamond w$$

is a valid formula. This formula establishes the connection between "necessity" and "possibility".

Another valid formula is

$$\Box(w_1 \supset w_2) \supset (\Box w_1 \supset \Box w_2),$$

i.e., if in all accessible states $w_1 \supset w_2$ holds and if w_1 is true in all accessible states, then w_2 must also be true in all of those states.

Both formulas are valid for any accessibility relation. If we agree to place further general restrictions on the relation R, we obtain additional valid formulas which are true for any model with a relation satisfying these restrictions. According to the different restrictions we may impose on R, we obtain different modal systems. *In our discussion we stipulate that R is always reflexive and transitive*; *i.e.*, we consider a formula to be *valid* iff it is true in all states of every universe with a reflexive and transitive accessibility relation.

For example, the formula

$$\Box w \supset w$$

is valid since it is true for every reflexive model. It claims for a state s that if all states accessible from s satisfy w, then w is satisfied by s itself. This is obvious since s is accessible from itself (by reflexivity).

The formula

$$\Diamond \Diamond w \supset \Diamond w,$$

which stands for $(\Diamond(\Diamond w)) \supset (\Diamond w)$, is valid since it is true for all transitive models. It claims for a state s_0: if there exists an s_2 accessible from s_1 which is accessible from s_0 such that s_2 satisfies w, then there exists an s_3 accessible from s_0 which satisifies w. This always holds in a transitive model since by transitivity, s_2 is also accessible from s_0 and we may take $s_3 = s_2$.

THE TEMPORAL FRAMEWORK

The framework of temporal logic is a modal framework in which we impose further restrictions on the models of interpretation ([PRI], [RU]). The interpretation given by temporal logic to the basic accessibility relation is that of the passage of time. A world s' is accessible from a world s if through development in time, s can change into s'. We concentrate on histories of development which are linear and discrete. Thus, the models of temporal logic consist of ω-sequences, *i.e.*, infinite sequences of the form $\sigma = s_0, s_1, \ldots$. In such a sequence, s_j is accessible from s_i *iff* $i \leq j$. Due to the discreteness of the sequences we can refer not only to states that lie in the future of a given state, but also to the (unique) immediate future state or *next state*. This leads to the introduction of an additional operator, the *next instant* operator denoted by $\bigcirc$.

Relating these concepts to the general modal framework, a universe for temporal logic consists again of a collection of states (worlds). On these states we define an *immediate accessibility relation* ρ which is required to be a function. That means that every world s has exactly one other world s' such that $\rho(s, s')$. This corresponds to our intuition that in a discrete time model each instant has exactly one immediate successor. $R = \rho^*$, the transitive reflexive closure of ρ, is the accessibility relation discussed under the general modal framework and is indeed both reflexive and transitive. Intuitively $R(s, s')$ holds when s' is either identical to s or lies in the future of s.

Given the restrictions imposed on R, the resulting model (U, s_0) can be represented as an *infinite sequence* of states,

$$\sigma = s_0,\ s_1,\ s_2,\ \ldots$$

where $\rho(s_i, s_{i+1})$ is true for $i \geq 0$. This intuitively corresponds to the temporal development of a process observed at a sequence of discrete points in time.

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We will now give a more complete definition of the language we are going to use. Note that this language is designed specially for the application we have in mind, namely reasoning about programs, and is not necessarily the most general temporal language possible.

Symbols. The language uses a set of basic symbols consisting of individual variables and constants, and proposition, function and predicate symbols. The set is partitioned into two subsets: global and local symbols. The *global symbols* have a uniform interpretation over the complete universe and do not change their value or meaning from one state to another. The *local symbols*, on the other hand, may assume different meanings and values in different states of the universe. For our purpose, the only local symbols that interest us are local individual variables and local propositions. We will have global symbols of all types.

Our symbols are further partitioned into different *sorts*. Each sort corresponds to a different domain, and the interpretation will associate a domain with every sort. Corresponding to a sort we may have individual constants that are interpreted over the associated domain, individual variables that assume values from that domain, function symbols that represent functions over the domain, and predicate symbols that represent predicates over the domain. The symbols used for individual constants, functions and predicates will be typical of the first-order theory of the domain we wish to formalize. For example, in dealing with the theory of natural numbers we use the conventional symbols:

$$\{0,\ 1,\ \ldots,\ +,\ -,\ \times,\ \div,\ \ldots,\ >,\ \geq,\ \ldots\}.$$

Note that some functions and predicates may have a non-homogenous *signature, i.e.*, they may have different sorts associated with different argument positions. A typical example is the *if-then-else* function which accepts one boolean argument and two arguments of possibly another sort.

Operators and quantifiers. We use the regular set of boolean connectives: $\wedge$, $\vee$, $\supset$, $\equiv$, and $\sim$ together with the equality operator $=$ and the first-order quantifiers $\forall$ and $\exists$. This set is referred to as the *classical operators.* The *modal operators* are:

$$\Box, \quad \Diamond, \quad \bigcirc \quad \text{and} \quad \mathcal{U};$$

they are called respectively the *always, sometime, next* and *until* operators. The first three operators are unary while the $\mathcal{U}$ operator is binary.

The quantifiers $\forall$ and $\exists$ are applied only to *global* individual variables.

Terms. Terms are constructed from individual constants and individual variables to which we apply functions. The application must conform with the arity and sort signature restrictions associated with each symbol. An additional rule is that if t is a term so is $\bigcirc t$ – referred to as the *next* (value of) t. Note that we use the *next* operator $\bigcirc$ in two different ways – as a temporal operator applied to formulas and as a temporal operator applied to terms.

Formulas (sentences). Formulas are constructed from atomic formulas to which we apply the boolean connectives, the modal operators and quantification over global individual variables. *Atomic formulas* consist of propositions and predicates (including the '$=$' operator) applied to terms of the appropriate sorts.

Recall that a formula is said to be *classical (static)* if it involves no modal operators.

We will sometimes regard propositions and (closed) formulas as integer-valued functions yielding 1 for *true* and 0 for *false.* These functions can then be combined arithmetically in order to provide a compact representation for equivalent but longer propositional formulas. For example, for propositions $p_1, \ldots, p_n$, the statement

$$p_1 + \cdots + p_n = 1 \quad \text{or} \quad \sum_{i=1}^{n} p_i = 1$$

states that exactly one of the p_i's is true. This is of course equivalent to the formula

$$\bigvee_{1 \le i \le n} p_i \quad \wedge \quad \bigwedge_{1 \le i < j \le n} \sim(p_i \wedge p_j).$$

MODELS (ENVIRONMENTS)

A *model* (I, α, σ) for our language consists of an (global) interpretation I, a (global) assignment α and a sequence of states σ.

The *interpretation* I specifies a nonempty domain D_i corresponding to each sort, and assigns concrete elements, functions and predicates to the (global) individual constants, function and predicate symbols.

The *assignment* α assigns a value over the appropriate domain to each of the global free individual variables.

The *sequence* $\sigma = s_0, s_1, \ldots$ is an infinite sequence of states. Each state s_i assigns values to the local free individual variables and propositions.

For a sequence

$$\sigma = s_0, s_1, \ldots$$

we denote by

$$\sigma^{(i)} = s_i, s_{i+1}, \ldots$$

the i-truncated suffix of σ.

Given a temporal formula w, we present below an inductive definition of the truth value of w in a model (I, α, σ). The value of a subformula or term τ under (I, α, σ) is denoted by $\tau|_{\sigma}^{\alpha}$, I being implicitly assumed.

Consider first the evaluation of terms:

- For a local individual variable or local proposition y:

$$y|_{\sigma}^{\alpha} = y_{s_0},$$

i.e., the value assigned to y in s_0, the first state of σ.

- For a global individual variable or global proposition u:

$$u|_{\sigma}^{\alpha} = \alpha[u],$$

i.e., the value assigned to u by α.

- For an individual constant the evaluation is given by I:

$$c|_{\sigma}^{\alpha} = I[c].$$

- For a k-ary function f:

$$f(t_1, \ldots, t_k)|_{\sigma}^{\alpha} = I[f](t_1|_{\sigma}^{\alpha}, \ldots, t_k|_{\sigma}^{\alpha}),$$

i.e., the value is given by the application of the interpreted function $I[f]$ to the values of $t_1, \ldots, t_k$ evaluated in the environment (I, α, σ).

- For a term t:

$$\bigcirc t|_{\sigma}^{\alpha} = t|_{\sigma^{(1)}}^{\alpha},$$

i.e., the value of $\bigcirc t$ in $\sigma = s_0, s_1, \ldots$ is given by the value of t in the shifted sequence $\sigma^{(1)} = s_1, s_2, \ldots$.

Consider now the evaluation of sentences:

- For a k-ary predicate p (including equality):

$$p(t_1, \ldots, t_k)|_\sigma^\alpha = I[p](t_1|_\sigma^\alpha, \ldots, t_k|_\sigma^\alpha).$$

Here again, we evaluate the arguments in the environment and then test $I[p]$ on them.

- For a disjunction:

$$(w_1 \vee w_2)|_\sigma^\alpha = \mathit{true} \quad \mathit{iff} \quad w_1|_\sigma^\alpha = \mathit{true} \quad \text{or} \quad w_2|_\sigma^\alpha = \mathit{true}.$$

- For a negation:

$$(\sim w)|_\sigma^\alpha = \mathit{true} \quad \mathit{iff} \quad w|_\sigma^\alpha = \mathit{false}.$$

- For a next-time application:

$$\bigcirc\, w|_\sigma^\alpha = w|_{\sigma^{(1)}}^\alpha.$$

Thus $\bigcirc\, w$ means: w will be true in the *next* instant – read "next w".

- For an all-times application:

$$\Box\, w|_\sigma^\alpha = \mathit{true} \quad \mathit{iff} \quad \text{for every } k \geq 0,\ w|_{\sigma^{(k)}}^\alpha = \mathit{true},$$

i.e., w is true for all suffix sequences of σ. Thus $\Box w$ means: w is true for *all* future instants (including the present) – read "always w" or "henceforth w".

- For a some-time application:

$$\Diamond\, w|_\sigma^\alpha = \mathit{true} \quad \mathit{iff} \quad \text{there exists a } k \geq 0 \text{ such that } w|_{\sigma^{(k)}}^\alpha = \mathit{true},$$

i.e., w is *true* on at least one suffix of σ. Thus $\Diamond\, w$ means: w will be true for *some* future instant (possibly the present) – read "sometimes w" or "eventually w".

- For an until application:

$$w_1\, \mathcal{U}\, w_2|_\sigma^\alpha = \mathit{true} \quad \mathit{iff} \quad \begin{array}{l}\text{for some } k \geq 0,\ w_2|_{\sigma^{(k)}}^\alpha = \mathit{true} \text{ and} \\ \text{for all } i,\ 0 \leq i < k,\ w_1|_{\sigma^{(i)}}^\alpha = \mathit{true}.\end{array}$$

Thus $w_1\, \mathcal{U}\, w_2$ means: there is a future instant in which w_2 holds, and such that *until* that instant w_1 continuously holds – read "w_1 until w_2"([KAM], [GPSS]).

- For a universal quantification:

$$(\forall u.w)|_\sigma^\alpha = \mathit{true} \quad \mathit{iff} \quad \text{for every } d \in D_i,\ w|_\sigma^{\alpha'} = \mathit{true},$$

where $\alpha' = \alpha \circ [u \leftarrow d]$ is the assignment obtained from α by assigning d to u. D_i is the domain corresponding to the sort of u.

- For an existential quantification:

$$(\exists u.w)|_{\sigma}^{\alpha} = \mathit{true} \quad \mathit{iff} \quad \text{for some } d \in D_i,\ w|_{\sigma}^{\alpha'} = \mathit{true},$$

where $\alpha' = \alpha \circ [u \leftarrow d]$.

Following are some examples of temporal expressions and their intuitive interpretations:

$u \supset \Diamond v$	—	If u is presently true, v will eventually become true.
$\Box(u \supset \Diamond v)$	—	Whenever u becomes true it will eventually be followed by v.
$\Diamond \Box w$	—	At some future instant w will become permanently true.
$\Diamond(w \wedge \bigcirc \sim w)$	—	There will be a future instant such that w is true at that instant and false at the next.
$\Box \Diamond w$	—	Every future instant is followed by a later one in which w is true, thus w is true infinitely often.
$\Box(u \supset \Box v)$	—	If u ever becomes true, then v is true at that instant and ever after.
$\Box u \vee (u\, \mathcal{U}\, v)$	—	Either u holds continuously or it holds until an occurrence of v. This is the weak form of the *until* operator that states that u will hold continuously until the first occurrence of v if v ever happens or indefinitely otherwise.
$\Diamond v \supset ((\sim v)\, \mathcal{U}\, u)$	—	If v ever happens, its first occurrence is preceded by (or coincides with) u.

If w is true under the model (I, α, σ) we say that (I, α, σ) *satisfies* w or that (I, α, σ) is a *satisfying model for* w. We denote this by

$$(I, \alpha, \sigma) \models w.$$

A formula w is *satisfiable* if there exists a satisfying model for it.

A formula w is *valid* if it is true in every model, and we write

$$\models w.$$

Sometimes we are interested in a restricted class of models C. A formula w which is true for every model in C is said to be *C-valid*, denoted by

$$C \models w.$$

Example:

The formula $\Diamond(w_1 \wedge w_2) \supset (\Diamond w_1 \wedge \Diamond w_2)$ is valid, *i.e.*,

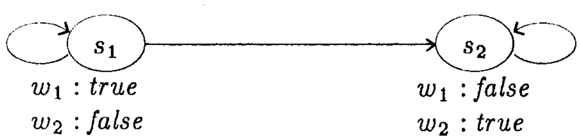
$$\models \Diamond(w_1 \wedge w_2) \supset (\Diamond w_1 \wedge \Diamond w_2).$$

It says that if there exists an instant in which both w_1 and w_2 are true then there exists an instant in which w_1 is true and there exists an instant in which w_2 is true.

The $\supset$-converse of this formula is not valid, *i.e.*,

$$\not\models \ (\Diamond w_1 \land \Diamond w_2) \ \supset \ \Diamond(w_1 \land w_2).$$

For, consider an interpretation in which w_1 is true and w_2 is false at state s_1, and in which w_1 is false and w_2 is true at state s_2, and s_2 is accessible from s_1 (also clearly s_1 from s_1 and s_2 from s_2)



Then at state s_1:

$\Diamond w_1$ is true (since w_1 is true at s_1)

$\Diamond w_2$ is true (since w_2 is true at s_2)

$\Diamond(w_1 \land w_2)$ is false (since $w_1 \land w_2$ is false at s_1 and at s_2).

Therefore, the formula is false under this interpretation. ∎

A REPERTOIRE OF VALID TEMPORAL STATEMENTS

In this section we present a list of valid temporal statements (schemata) which we justify by semantic considerations. There are two reasons for presenting them here. First we would like to illustrate the type of temporal reasoning we will later use. Second, the statements presented here will later be taken to be established valid statements and used freely in proofs. When, in a later part of this work, we present a formal deductive system for temporal reasoning, we will take some of the valid statements listed here as axioms and deduce the others as theorems.

In the following list, whenever we write a valid temporal statement in form $\models A \supset B$ and not $\models A \equiv B$, it implies that its $\supset$-inverse is not valid, *i.e.*, $\not\models \ B \supset A$. That is, a model can be found under which an instance of $B \supset A$ will be false.

1. $\models \ \Box \sim w \ \equiv \ \sim \Diamond w$

2. $\models \ \Diamond \sim w \ \equiv \ \sim \Box w$

3. $\models \ \bigcirc \sim w \ \equiv \ \sim \bigcirc w$

These statements point out the *duality* between the operators.

Statement 1 says that w is false in all states (instants) of a sequence *iff* there is no state in which w is true.

Statement 2 says that there is a state in which w is false *iff* it is not the case that w is true in all states.

Statement 3 says that w is false in the next state *iff* it is not the case that w is true in the next state. This statement restricts each state to have a single successor.

4. $\models w \supset \Diamond w$

5. $\models \Box w \supset w$

6. $\models \bigcirc w \supset \Diamond w$

7. $\models \Box w \supset \bigcirc w$

8. $\models \Box w \supset \Diamond w$

9. $\models \Box w \supset \bigcirc \Box w$

10. $\models w_1 \mathcal{U} w_2 \supset \Diamond w_2$

11. $\models \Diamond \Box w \supset \Box \Diamond w.$

Statement 4 says that if w is true now, then it will be true sometime in the future. This is an immediate consequence of the fact that the present is considered to be part of the future.

Statement 5, a dual of 4, says that if w is true in all future instants it is also presently true.

Statement 6 says that if w is true at the next instant it will sometime be true. This is because the next instant is also a part of the future.

Statement 7, a dual of 6, says that if w is true in all future instants it is also true for the next instant.

Statement 8 says that if w is always true then it is sometimes true.

Statement 9 says that if w is true in all future instants it is also true for all future instants of the next instant, *i.e.*, all future instants excluding the present.

Statement 10 says that if w_1 is true until w_2 will happen then w_2 will eventually happen.

Statement 11 says that if w is permanently true beyond a certain instant then w is true infinitely often.

12. $\models \Box w \equiv \Box \Box w$

13. $\models \Diamond w \equiv \Diamond \Diamond w.$

The statements 12 and 13 say that both $\Box$ and $\Diamond$ are *idempotent*. Intuitively speaking both imply that the future is equivalent to the future of the future. Note that a corresponding statement does not hold for $\bigcirc$, since both $\not\models \bigcirc w \supset \bigcirc \bigcirc w$ and $\not\models \bigcirc \bigcirc w \supset \bigcirc w$.

14. $\models \Box \bigcirc w \equiv \bigcirc \Box w$

15. $\models \Diamond \bigcirc w \equiv \bigcirc \Diamond w$

$$16. \quad \models \quad ((\bigcirc w_1)\,\mathcal{U}\,(\bigcirc w_2)) \;\equiv\; \bigcirc(w_1\,\mathcal{U}\,w_2).$$

Statements 14 to 16 indicate the *commutativity* of the *next* operator $\bigcirc$ with each of the others. It amounts to a shift of our reference point from the present to the immediately next instant.

Statement 14 says that w holds for the instant next to every future instant *iff* w holds for all future instants, barring the present.

Statement 15 says that w is realized in an instant next to some future instant *iff* it is realized sometimes in the future, excluding the present.

Statement 16 says that $\bigcirc w_1$ holds until an instance of $\bigcirc w_2$ *iff* w_1 holds until w_2 starting from the next instant.

$$17. \quad \models \quad \Box(w_1 \wedge w_2) \;\equiv\; (\Box w_1 \wedge \Box w_2)$$
$$18. \quad \models \quad \Diamond(w_1 \vee w_2) \;\equiv\; (\Diamond w_1 \vee \Diamond w_2)$$
$$19. \quad \models \quad \bigcirc(w_1 \wedge w_2) \;\equiv\; (\bigcirc w_1 \wedge \bigcirc w_2)$$
$$20. \quad \models \quad \bigcirc(w_1 \vee w_2) \;\equiv\; (\bigcirc w_1 \vee \bigcirc w_2)$$
$$21. \quad \models \quad \bigcirc(w_1 \supset w_2) \;\equiv\; (\bigcirc w_1 \supset \bigcirc w_2)$$
$$22. \quad \models \quad \bigcirc(w_1 \equiv w_2) \;\equiv\; (\bigcirc w_1 \equiv \bigcirc w_2)$$
$$23. \quad \models \quad ((w_1 \wedge w_2)\,\mathcal{U}\,w_3) \;\equiv\; ((w_1\,\mathcal{U}\,w_3) \wedge (w_2\,\mathcal{U}\,w_3))$$
$$24. \quad \models \quad (w_1\,\mathcal{U}\,(w_2 \vee w_3)) \;\equiv\; ((w_1\,\mathcal{U}\,w_2) \vee (w_1\,\mathcal{U}\,w_3)).$$

Statements 17 to 24 indicate *distributivity* relations between the temporal operators and the boolean connectives.

The $\Box$ operator has a *universal* character – stating w for *all* future instants, and the $\Diamond$ operator has an *existential* character – stating w for *some* future instant. Consequently $\Box$ distributes with $\wedge$ (17) stating that both w_1 and w_2 hold in every future instant *iff* w_1 holds for all future instants and so does w_2. The $\Diamond$ operator distributes with $\vee$ (18) stating that there will be an instant in which either w_1 or w_2 hold *iff* there either will be an instant in which w_1 holds or there will be an instant in which w_2 holds.

The $\bigcirc$ operator has both universal and existential character because it refers to a unique instant – the next one. Therefore it distributes with both $\wedge$ and $\vee$, as is shown by statements 19 and 20.

Since the $\bigcirc$ operator has been shown to distribute with the basic boolean connectives $\sim$, $\wedge$, $\vee$, it will also distribute over any other boolean connective such as $\supset$ and $\equiv$. For example, Statement 21 says that if in the next instant w_1 implies w_2 and w_1 is known to hold at the next instant then so does w_2.

The *until* operator has a different character with respect to its two arguments. It is universal with respect to its first argument which appears in the semantic definition under a $\forall i(0 \leq i < k)$ quantification. It is existential with respect to its second argument which appears in the semantic definition under a $\exists k(k \geq 0)$ quantification.

Statement 23 says that w_1 and w_2 both hold until an instance of w_3 *iff* w_1 holds until an instance of w_3 and w_2 holds until an instance of w_3. To justify the implication from right to left, we are guaranteed of having a t_1 such that w_3 is true at t_1 and w_1 holds until then, and a t_2 such that w_3 is true at t_2 and w_2 holds until then. By considering the earliest of these two instants $t = min(t_1, t_2)$ we know that w_3 is true at t and both w_1 and w_2 hold until then.

Statement 24 says that w_1 holds until an instance of either w_2 or w_3 *iff* either w_1 holds until an instance of w_2 or w_1 holds until an instance of w_3.

25. $\models \; (\Box w_1 \vee \Box w_2) \;\; \supset \;\; \Box(w_1 \vee w_2)$

26. $\models \; \Diamond(w_1 \wedge w_2) \;\; \supset \;\; (\Diamond w_1 \wedge \Diamond w_2)$

27. $\models \; ((w_1 \, \mathcal{U} \, w_3) \vee (w_2 \, \mathcal{U} \, w_3)) \;\; \supset \;\; (w_1 \vee w_2) \, \mathcal{U} \, w_3$

28. $\models \; (w_1 \, \mathcal{U} \, (w_2 \wedge w_3)) \;\; \supset \;\; ((w_1 \, \mathcal{U} \, w_2) \wedge (w_1 \, \mathcal{U} \, w_3)).$

Statements 25 to 28 indicate *implications* that hold when we interchange the order between temporal operators and the boolean connectives. They are not equivalences and only the direction of the given implication is true.

Statement 25 says that if either w_1 is true for all future instants or w_2 is true for all future instants then in every future instant either w_1 or w_2 holds.

Statement 26 says that if there exists an instant in which both w_1 and w_2 are true then there exists an instant in which w_1 is true and there exists an instant in which w_2 is true.

Statement 27 says that if either w_1 holds until w_3 or w_2 holds until w_3 then there is an instance of w_3 such that until then either w_1 or w_2 holds.

Statement 28 says that if w_1 holds until an instant t in which both w_2 and w_3 are true then both w_1 holds until w_2 at t and w_1 holds until w_3 at t implying the conjunction.

29. $\models \; \Box(w_1 \supset w_2) \;\; \supset \;\; (\Box w_1 \supset \Box w_2)$

30. $\models \; \Box(w_1 \supset w_2) \;\; \supset \;\; (\Diamond w_1 \supset \Diamond w_2)$

31. $\models \; \Box(w_1 \supset w_2) \;\; \supset \;\; (\bigcirc w_1 \supset \bigcirc w_2)$

32. $\models \; \Box(w_1 \supset w_2) \;\; \supset \;\; ((w_1 \, \mathcal{U} \, w_3) \supset (w_2 \, \mathcal{U} \, w_3))$

33. $\models \; \Box(w_1 \supset w_2) \;\; \supset \;\; ((w_0 \, \mathcal{U} \, w_1) \supset (w_0 \, \mathcal{U} \, w_2)).$

Statements 29 to 33 indicate the *monotonicity* of each of the temporal operators; that is, if its application to a formula w_1 is true and w_1 universally implies w_2 (for all instants) then its application to w_2 is also true.

This property is stated respectively for $\Box$ in 29, $\Diamond$ in 30, $\bigcirc$ in 31 and the two positions of $\mathcal{U}$ in 32 and 33.

34. $\models \; (\Box w_1 \wedge \bigcirc w_2) \;\; \supset \;\; \bigcirc(w_1 \wedge w_2)$

35. $\models \; (\Box w_1 \wedge \Diamond w_2) \;\; \supset \;\; \Diamond(w_1 \wedge w_2)$

36. $\models$ $(\Box w_1 \land (w_2 \,\mathcal{U}\, w_3)) \supset (w_1 \land w_2) \,\mathcal{U}\, (w_1 \land w_3)$.

Statements 34 to 36 are *frame rules.* They say that if w_1 is known to hold for all states then w_1 may be added as a conjunct under any other temporal operator. This is respectively stated for $\bigcirc$ in 34, for $\Diamond$ in 35 and for both argument positions of $\mathcal{U}$ in 36.

37. $\models$ $(w \land \Box(w \supset \bigcirc w)) \supset \Box w$

38. $\models$ $(w \land \Diamond \sim w) \supset \Diamond(w \land \bigcirc \sim w)$.

39. $\models$ $(\Diamond w_1 \land \Diamond w_2) \supset [\Diamond(w_1 \land \Diamond w_2) \lor \Diamond(w_2 \land \Diamond w_1)]$.

Statements 37 and 38 are *induction rules* and Statement 39 describes the *linearity property.*

Statement 37 (corresponding to *computational induction*) says that if the fact that w holds at any instant implies that it also holds at the next instant, and w holds in the present, then w holds at all future instants.

Statement 38 (corresponding to *the least number principle*) is the dual of 37. It says that if w is true now and is false sometime in the future, then there exists some instant such that w is true at that instant and false at the next.

Statement 39 says that if w_1 and w_2 are both guaranteed to happen, then either w_1 will happen first, followed by w_2 or w_2 will happen first, followed by w_1.

40. $\models$ $\Box w \equiv (w \land \bigcirc \Box w)$

41. $\models$ $\Diamond w \equiv (w \lor \bigcirc \Diamond w)$

42. $\models$ $w_1 \,\mathcal{U}\, w_2 \equiv w_2 \lor (w_1 \land \bigcirc(w_1 \,\mathcal{U}\, w_2))$

Statements 40 to 42 explain the $\Box$, $\Diamond$, and $\mathcal{U}$ operators respectively by distributing their effect into what is implied for the present and what is implied for the next instant.

Statement 40 says that w is true for all future instants *iff* w is true for the present and for all instants lying in the future of the next instant.

Statement 41 says that w is true in some future instance *iff* it is either true now or true at an instant not earlier than the next.

Statement 42 says that 'w_1 until w_2' is presently true *iff* either w_2 is true now or w_1 holds now and 'w_1 until w_2' is true for the next instant.

43. $\models$ $(\sim w \,\mathcal{U}\, w) \equiv \Diamond w$

44. $\models$ $(\Box w_1 \land \Diamond w_2) \supset (w_1 \,\mathcal{U}\, w_2)$

45. $\models$ $((w_1 \supset w_2) \,\mathcal{U}\, w_3) \supset ((w_1 \,\mathcal{U}\, w_3) \supset (w_2 \,\mathcal{U}\, w_3))$

46. $\models$ $((w_1 \,\mathcal{U}\, w_2) \land (\sim w_2 \,\mathcal{U}\, w_3)) \supset (w_1 \,\mathcal{U}\, w_3)$

47. $\models$ $(w_1 \,\mathcal{U}\, (w_2 \land w_3)) \supset ((w_1 \,\mathcal{U}\, w_2) \,\mathcal{U}\, w_3)$

48. $\models$ $((w_1 \,\mathcal{U}\, w_2) \,\mathcal{U}\, w_3) \;\supset\; ((w_1 \vee w_2) \,\mathcal{U}\, w_3)$

49. $\models$ $(\Diamond w_1 \wedge \Diamond w_2) \;\supset\; ((\sim w_1 \,\mathcal{U}\, w_2) \vee (\sim w_2 \,\mathcal{U}\, w_1))$.

This list of statements illustrates some properties of the *until* operator.

Statement 43 says that w is guaranteed to happen *iff* there is an instant in which w is true and until this instant w is false. This states that w happens *iff* there is an earliest occurence of w.

Statement 44 says that if w_2 is guaranteed to happen and w_1 is constantly true, then it will be true until a guaranteed occurence of w_2.

Statement 45 says that if w_1 implies w_2 until w_3 happens and w_1 is true until an instance of w_3 (not necessarily the same instance) then w_2 will hold until an instance of w_3 (which can be taken as the earlier of the two).

Statement 46 says that if w_1 holds until w_2 and w_2 is false until w_3 then w_1 is true until w_3. To justify this let (a) $w_1 \,\mathcal{U}\, w_2$ and (b) $\sim w_2 \,\mathcal{U}\, w_3$ be the two clauses given as premises. By (b) we know that w_3 will happen say at t_3 and w_2 will be false until then. By (a) w_2 must happen, say at t_2 and w_1 must be true until then. By (b) $t_2 \geq t_3$ so that w_1 must certainly be true until t_3, an instance of w_3.

Statement 47 can be justified as follows. The premise guarantees an instant t_2 such that w_2 and w_3 are both true at t_2 and w_1 is true until then. Clearly, taking any $0 \leq t_1 < t_2$ we know that w_2 will be true at t_2 and w_1 is true for every t, $t_1 \leq t < t_2$, thus $w_1 \,\mathcal{U}\, w_2$ at t_1. Since $w_1 \,\mathcal{U}\, w_2$ is true for every t_1, $0 \leq t_1 < t_2$, and w_3 is true at t_2, $w_1 \,\mathcal{U}\, w_2$ is true until w_3.

Statement 48 says that if $w_1 \mathcal{U} w_2$ is continuously true until an instance of w_3 then so is $w_1 \vee w_2$.

Statement 49 says that if both w_1 and w_2 are guaranteed to happen then one of them will happen "first"; that is, either w_2 happens first and w_1 is false until then, or w_1 happens first and w_2 is false until then. (In both cases we allow the possibility that both w_1 and w_2 occur for the first time at the same instant.)

50. $\models$ $\Diamond \exists x w \;\equiv\; \exists x \Diamond w$

51. $\models$ $\Box \forall x w \;\equiv\; \forall x \Box w$

52. $\models$ $\bigcirc \exists x w \;\equiv\; \exists x \bigcirc w$

53. $\models$ $\bigcirc \forall x w \;\equiv\; \forall x \bigcirc w$

54. $\models$ $((\forall x w_1) \,\mathcal{U}\, w_2) \;\equiv\; \forall x (w_1 \,\mathcal{U}\, w_2)$ provided x is not free in w_2

55. $\models$ $(w_1 \,\mathcal{U}\, (\exists x w_2)) \;\equiv\; \exists x (w_1 \,\mathcal{U}\, w_2)$ provided x is not free in w_1

Statements 50 to 55 indicate the *commutativity* relations between the temporal operators and the quantifiers. They follow from our restriction that the quantifiers $\forall$ and $\exists$ are to be applied only to *global* individual variables. Statements 50 and 51 are known as *Barcan's formulas*.

Statement 50 demonstrates once more the existential character of the operator $\Diamond$. It says that in some instant there exists an x satisfying $w(x)$ *iff* there exists an x such that at some instant $w(x)$ is satisfied.

Statement 51 demonstrates the universal character of the $\Box$ operator. It says that w is true in all instants for all values of x *iff* it is true for all values of x for every instant.

Statements 52 and 53 demonstrate the dual character of the $\bigcirc$ operator, which is both universal and existential.

Statements 54 and 55 demonstrate that the *until* operator has a universal character with respect to its first argument and an existential character with respect to its second argument.

The preceeding statements were all of the form

$$\models w$$

and they stated *formulas* which are true in every model. The next list of statements contains *inferences* of the form

$$\models w_1 \Rightarrow \models w_2.$$

They state that if w_1 has been shown to be a valid statement then so is w_2. The inference statements enable us to deduce the validity of one formula from the other. For every valid formula $\models w_1 \supset w_2$ there is a corresponding inference $\models w_1 \Rightarrow \models w_2$, and this is a standard way of justifying an inference. However, there are inferences $\models w_1 \Rightarrow \models w_2$ such that $\models w_1 \supset w_2$ is not a valid statement (see, for example, the following inference 56).

56. $\models w \Rightarrow \models \Box w$ — $\Box$-*insertion*

57. $\models w \Rightarrow \models \Diamond w$ — $\Diamond$-*insertion*

58. $\models w \Rightarrow \models \bigcirc w$ — $\bigcirc$-*insertion*

Inference 56 states that if w is valid then so is $\Box w$. The fact that w is valid means that it is true for every sequence and therefore for all suffixes $\sigma^{(i)}$ of a given sequence. Thus $\Box w$ is true for every sequence σ and is therefore a valid statement.

Inference 57 may be deduced by inferring first $\models \Box w$ and then using the valid statement $\models \Box w \supset \Diamond w$ (number 8 in our list) to infer $\models \Diamond w$.

Inference 58 may be deduced similarly by using Statement 7, $\models \Box w \supset \bigcirc w$.

59. $\models w_1 \supset w_2 \Rightarrow \models \Box w_1 \supset \Box w_2$ — $\Box\Box$ -*insertion*

60. $\models w_1 \supset w_2 \Rightarrow \models \Diamond w_1 \supset \Diamond w_2$ — $\Diamond\Diamond$ -*insertion*

61. $\models w_1 \supset w_2 \Rightarrow \models \bigcirc w_1 \supset \bigcirc w_2$ — $\bigcirc\bigcirc$ -*insertion*

These inferences are all obtained by infering first $\models \Box(w_1 \supset w_2)$ by Inference 56 and then using statements 29 to 31, respectively.

62. $\left.\begin{array}{l}\models w_1 \supset \Box w_2 \\ \models w_2 \supset \Box w_3\end{array}\right\} \Rightarrow \models w_1 \supset \Box w_3$ — $\Box$-*concatenation*

63. $\left.\begin{array}{l}\models w_1 \supset \Diamond w_2 \\ \models w_2 \supset \Diamond w_3\end{array}\right\} \Rightarrow \models w_1 \supset \Diamond w_3$ — $\Diamond$-*concatenation*

Inference 62 is obtained by first deriving $\models \Box w_2 \supset \Box\Box w_3$ by Inference 59, observing that $\Box\Box w_3 \equiv \Box w_3$, and then using propositional reasoning. Inference 63 is obtained similarly by applying Inference 60. Note that the corresponding O-concatenation inference does not hold.

$$64. \quad \left.\begin{array}{l} \models w_1 \supset w_2 \\ \models w_2 \supset \Box w_3 \\ \models w_3 \supset w_4 \end{array}\right\} \Rightarrow \models w_1 \supset \Box w_4 \qquad \Box\text{–}\textit{consequence}$$

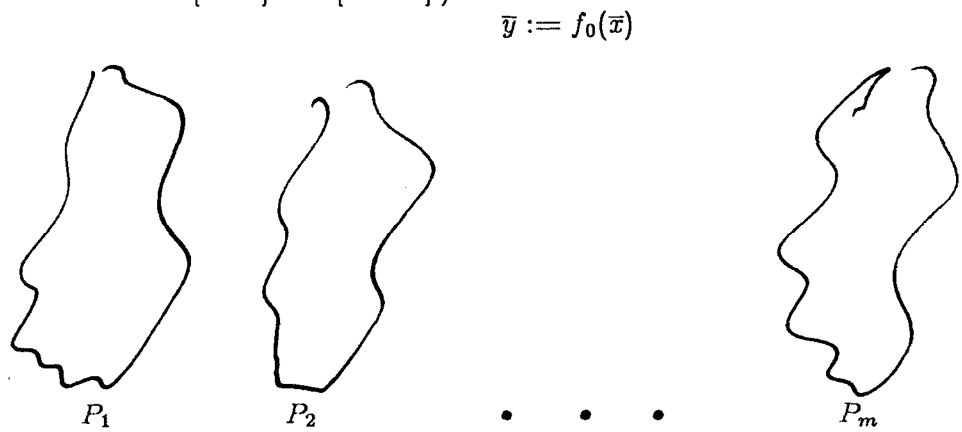
$$65. \quad \left.\begin{array}{l} \models w_1 \supset w_2 \\ \models w_2 \supset \Diamond w_3 \\ \models w_3 \supset w_4 \end{array}\right\} \Rightarrow \models w_1 \supset \Diamond w_4 \qquad \Diamond\text{–}\textit{consequence}$$

$$66. \quad \left.\begin{array}{l} \models w_1 \supset w_2 \\ \models w_2 \supset \bigcirc w_3 \\ \models w_3 \supset w_4 \end{array}\right\} \Rightarrow \models w_1 \supset \bigcirc w_4 \qquad \bigcirc\text{–}\textit{consequence}$$

Inference 64 is obtained by deriving first $\models \Box w_3 \supset \Box w_4$ by $\Box\Box$–introduction (59) and then applying propositional reasoning. Similarly, inferences 65 and 66 are obtained by deriving $\models \Diamond w_3 \supset \Diamond w_4$ and $\models \bigcirc w_3 \supset \bigcirc w_4$ by 60 and 61, respectively.

2. CONCURRENT PROGRAMS AND THEIR EXECUTION

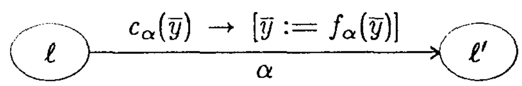
In the following we introduce the model of concurrent programs that we will study here. (For simpler models see [KEL] and [LAM1].)



In our model, a concurrent program

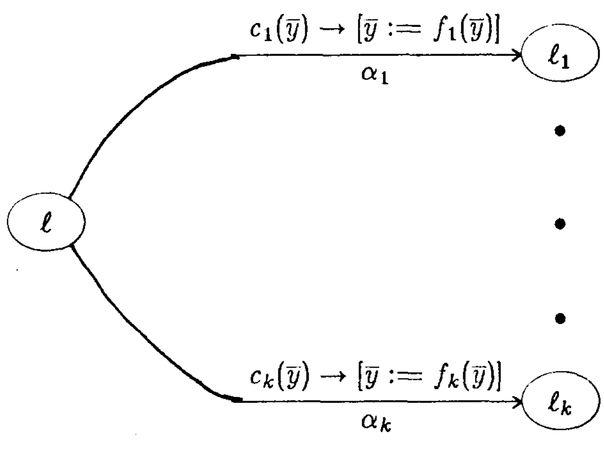
$$\overline{y} := f_0(\overline{x});\ [P_1 \| \ldots \| P_m]$$

consists of an initial value assignment $\overline{y} := f_0(\overline{x})$ followed by the parallel execution of m, $m \geq 1$, processes $P_1, \ldots, P_m$. The processes operate on a set of program variables $\overline{y} = (y_1, \ldots, y_n)$ which are shared between the processes. The variables $\overline{y}$ are accessible to all the processes for both referencing and modifying. Each process P_i, $i = 1, \ldots, m$, is an independent transition graph with nodes (locations) labeled by $\ell_0^i, \ell_1^i, \ldots, \ell_e^i$. The sets of labels $L_i = \{\ell_0^i, \ldots, \ell_e^i\}$ of the different processes are disjoint. The edges (or transitions) in each process are labeled by instructions of the form:



where $c_\alpha(\overline{y})$ is a condition called the *enabling condition* of the transition α, and f_α is the *transformation* associated with the transition α. If $c_\alpha(\overline{\eta})$ is true we say that the *transition* α *is enabled* for $\overline{y} = \overline{\eta}$.

For a given node ℓ with k outgoing transitions



we define $E_\ell(\overline{y}) = c_1(\overline{y}) \vee \ldots \vee c_k(\overline{y})$ to be the *full-exit condition* at node ℓ. We do not require that the individual conditions are exhaustive, *i.e.*, that $E_\ell(\overline{y}) =$ *true* for every $\overline{y}$; thus, *deadlocks* (or *blockings*) are allowed in our semantics. Nor do we require the conditions to be exclusive; thus, each process can be nondeterministic. A location whose individual conditions are mutually exclusive is called a *deterministic location.* If $E_\ell(\overline{\eta})$ is true, *i.e.* at least one of the α_i, $i = 1, \ldots, k$, transitions originating from ℓ is enabled, we say that the *location* ℓ *is enabled* for $\overline{y} = \overline{\eta}$. If a process P_j is currently at $\ell \in L_j$ which is enabled, we say that the *process is enabled.*

The set of program variables $\overline{y} = (y_1, \ldots, y_n)$ is accessible and shared by all the processes. This model of concurrent programs is therefore called the *shared-variables model.* In this model, communication and synchronization between processes are managed via the shared variables.

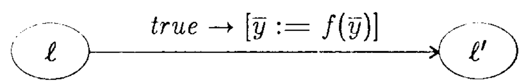
The initial assignment $\overline{y} := f_0(\overline{x})$ assigns initial values to the shared program variables prior to the beginning of the concurrent execution. The parameters $\overline{x} = (x_1, \ldots, x_t)$ that appear in this initial assignment, as well as other parameters appearing in the bodies of the processes, are the *input parameters* of the program. The behavior of the program naturally depends on the input parameters.

We will often represent a process in a *linear-text* form instead of a graph. In such a case the nodes are the places (labels) just before each statement, and the transitions are the statements themselves.

We list below the types of statements that we allow in the linear-text form and their representation in the graph model:

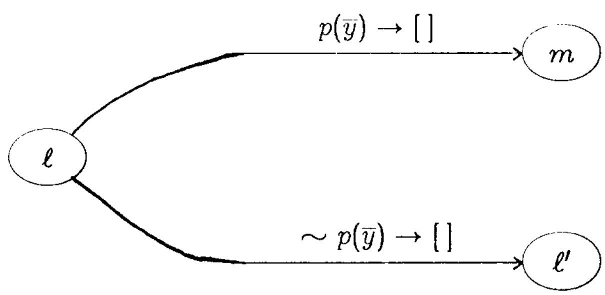
- ℓ : $\overline{y} := f(\overline{y})$
  ℓ' :

is represented as



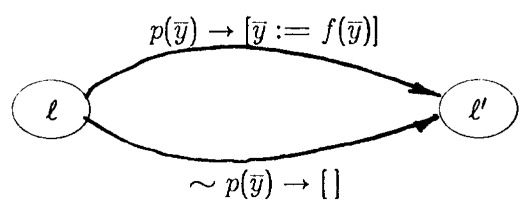
- ℓ : *if* $p(\overline{y})$ *then go to* m
  ℓ' :

is represented as



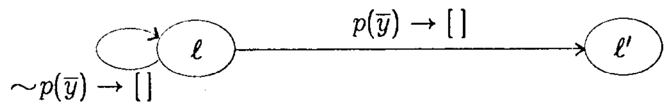
- ℓ : *if* $p(\overline{y})$ *then* $\overline{y} := f(\overline{y})$
  ℓ' :

is represented as



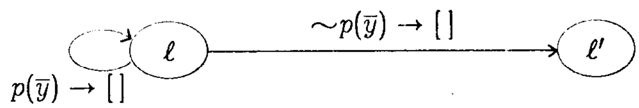
▶ ℓ : *loop until* $p(\overline{y})$
ℓ' :

This statement loops until the condition $p(\overline{y})$ becomes true. It is represented as



▶ ℓ : *loop while* $p(\overline{y})$
ℓ' :

This statement is the complement of the above statement: it loops until condition $p(\overline{y})$ is false. It is represented as



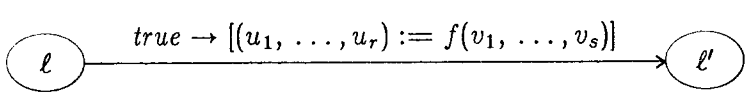
▶ ℓ : *compute* $u_1, \ldots, u_r$ *using* $v_1, \ldots, v_s$
ℓ' :

This statement represents a segment of *terminating computation* in whose details we are not interested. The only facts we assume about this segment are:

1. The segment may modify only the program variables $u_1, \ldots, u_r$, $r \geq 0$, and may reference only the program variables $v_1, \ldots, v_s$, $s \geq 0$.

2. The segment must eventually terminate.

The statement is represented as



where f represents an unspecified function.

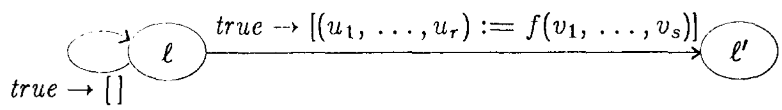
We will often use *compute* segments of the form

ℓ : *compute*
ℓ' :

for the case $r = s = 0$ to refer to a segment of terminating computation that does not modify or access any program variables.

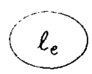
▸ ℓ : *execute* $u_1, \ldots, u_r$ *using* $v_1, \ldots, v_s$
ℓ' :

This statement represents an *arbitrary* program segment that may modify only the program variables $u_1, \ldots, u_r$, $r \geq 0$, and may reference only $v_1, \ldots, v_s$, $s \geq 0$. Here we do not require that the segment must eventually terminate. Consequently its representation is given by:



▸ ℓ_e : *halt*

is represented as:



i.e., a node with no exits.

Note that for all the statements considered so far, except for the *halt* statement, the *full-exit condition* is always identically true. Also all the instructions (and their corresponding locations), except for the *execute* $u_1, \ldots, u_r$ instruction, are deterministic, *i.e.*, they have mutually exclusive transitions.

Example:

Consider the following concurrent program for computing the binomial coefficient $\binom{n}{k}$ for integers n and k, such that $0 \leq k \leq n$:

Program BC (Binomial Coefficient):

$$y_1 := n, \quad y_2 := 0, \quad y_3 := 1$$

ℓ_0 : *if* $y_1 = (n-k)$ *then go to* ℓ_e	m_0 : *if* $y_2 = k$ *then go to* m_e
ℓ_1 : $y_3 := y_3 \cdot y_1$	m_1 : $y_2 := y_2 + 1$
ℓ_2 : $y_1 := y_1 - 1$	m_2 : *loop until* $y_1 + y_2 \leq n$
ℓ_3 : *go to* ℓ_0	m_3 : $y_3 := y_3 / y_2$
ℓ_e : *halt*	m_4 : *go to* m_0
	m_e : *halt*
— *Process* P_1 —	— *Process* P_2 —

The input parameters to this program are n and k. Note that n appears in the initial assignment while both n and k appear in statements of the processes.

We have not yet discussed the execution of concurrent programs in our model. Assume for a moment that each instruction in this program is atomic and that at any instant only one such atomic instruction is executed. Once it is completed, another instruction (from either process) is executed to its completion, and so on. Under this assumption, the program BC_0 computes the binomial coefficient

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \cdots \cdot (n-k+1)}{1 \cdot 2 \cdot \cdots \cdot k}$$

The values of y_1, *i.e.*, n, $n-1$, $\ldots$, $n-k+1$, are used to compute the numerator in P_1 (the last value of y, $n-k$, is not used), and the values of y_2, *i.e.*, 1, 2, $\ldots$, k, are used to compute the denominator (the first value of y_2, 0, is not used). The process P_1 multiplies $n \cdot (n-1) \cdot \cdots \cdot (n-k+1)$ into y_3 while P_2 divides y_3 by $1 \cdot 2 \cdot \cdots \cdot k$.

The instruction

$$m_2: \quad \textit{loop until } y_1 + y_2 \leq n$$

guarantees even divisibility. It synchronizes P_2's operation with that of P_1 to ensure that y_3 is divided by i only after it has been multiplied by $n-i+1$. We rely here on the mathematical theorem that the product of i consecutive positive integers: $k \cdot (k+1) \cdot \cdots \cdot (k+i-1)$ is always divisible by $i!$.

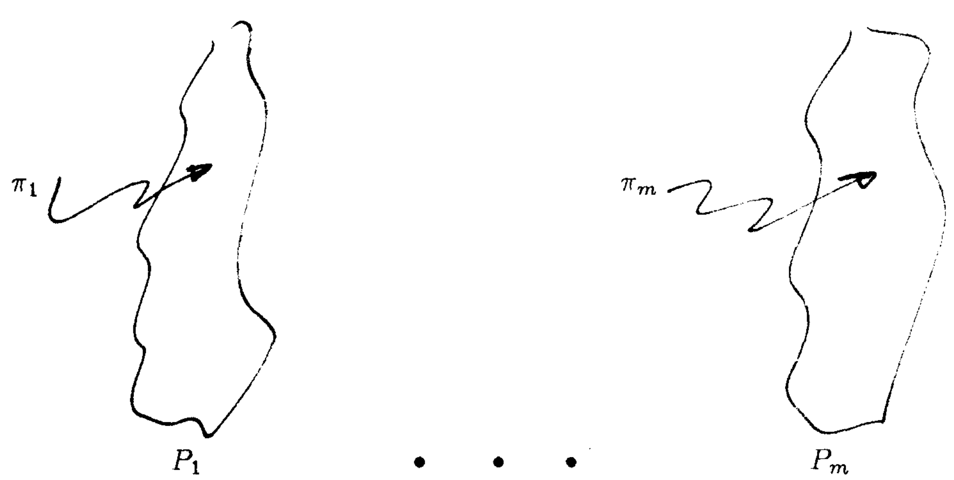
Now, consider the intermediate expression at m_2:

$$y_3 = \frac{n \cdot (n-1) \cdot \cdots \cdot (n-j+1)}{1 \cdot 2 \cdot \cdots \cdot (i-1)},$$

where $1 \leq i \leq j \leq n$, $y_1 = n-j$ and $y_2 = i$. The numerator consists of the product of j consecutive positive integers and is therefore divisible by i since $i \leq j$. If $j = i$, we have to wait until y_1 is decremented by the instruction in ℓ_2 from $n-i+1$ to $n-i$ before we can be absolutely sure that $(n-i+1)$ has been multiplied into y_3. Thus, process P_2 waits at m_2 until $y_1 + y_2$ drops to a value less than or equal to n. ∎

In order to keep track of the progress of the execution in each process we use a vector of *location variables* $\overline{\pi} = \{\pi_1, \ldots, \pi_m\}$ where each π_i ranges over the label set L_i of process P_i.

$$y := f_0(\overline{x})$$



The location variable π_i points to the location in P_i which is to be executed next.

CONCURRENCY AND ITS MODELLING BY INTERLEAVING

Before defining the execution of concurrent programs in our model, we should first study in more detail the actual behavior of a physically concurrent system.

As our motivating real-life situation we consider a system consisting of m physically separate processors $\Pi_1, \ldots, \Pi_m$. Each of the processors Π_i is responsible for executing the process program P_i. The shared program variables $y_1, \ldots, y_n$ reside in a common memory M to which each of the processors must gain access in order to retrieve or store a value of a shared variable. In addition, each of the processors has its own set of *private variables (registers).* These are used to hold intermediate results of the computation or values which are not needed by the other processes. We will refer to these private registers as $t_0, t_1, \ldots$. We assume that the shared memory, M, is hardware protected to allow only one processor to access a shared variable at a certain instant. While the access is taking place, the particular variable accessed is unavailable to all other processors. Each access is restricted to a single operation, a value retrieval or a value update, but not both.

Consider for example the joint operation of two processors Π_1 and Π_2 which are executing the following concurrent program:

Elementary Program EP

$$y := 0$$

$\ell_0: \; t_1 := y$	$m_0: \; t_2 := y$
$\ell_1: \; t_1 := t_1 - 1$	$m_1: \; t_2 := t_2 + 1$
$\ell_2: \; y := t_1$	$m_2: \; y := t_2$
$\ell_e: \; halt$	$m_e: \; halt$
$— P_1 —$	$— P_2 —$

Each processor Π_i has its private register t_i, $i = 1, 2$. This program has been carefully constructed so that it uses only three standardized types of *elementary instructions*:

a. A *shared retrieval (reference)*, transferring the current value of a shared variable into a private register:

$$t_1 := y \quad \text{and} \quad t_2 := y.$$

b. A *shared update (modification)*, storing the value of a private register into a shared variable:

$$y := t_1 \quad \text{and} \quad y := t_2.$$

c. An *internal computation* of the form $t_i := f(\bar{t})$ assigning to one register of a processor a value which is a function of the registers $\bar{t}$ of the same processor:

$$t_1 := t_1 - 1 \quad \text{and} \quad t_2 := t_2 + 1.$$

We also frequently use a fourth type of elementary instruction:

d. An *internal test* of the form

$$\textit{if } p(\bar{t}) \textit{ then go to } \ell,$$

where $\bar{t}$ are registers of the same processor.

With the execution of the instructions of types a and b we can associate a unique event which is the actual access to the shared memory M. We refer to these events as *shared access events*. For the simple program presented above we can associate the events r_i, $i = 1, 2$, with the *retrieval* of the value of the shared y at the instruction in locations ℓ_0 and m_0 respectively. Similarly, we associate the events $u_i, i = 1, 2$, with the *updating* of the shared variable y at the instructions ℓ_2 and m_2 respectively. No access event is associated with internal computations such as those at ℓ_1 and m_1.

Since in our example all four accesses refer to the same variable y, no two of them can occur exactly at the same time because of the exclusivity mechanism provided by the memory unit M. Thus in any possible concurrent execution of this program we will observe a linear sequence of the occurrences of these four events. The only possible sequences are:

r_1, u_1, r_2, u_2	leading to a final value of	$y = 0$
r_2, u_2, r_1, u_1	leading to a final value of	$y = 0$
r_1, r_2, u_1, u_2	leading to a final value of	$y = 1$
r_1, r_2, u_2, u_1	leading to a final value of	$y = -1$
r_2, r_1, u_1, u_2	leading to a final value of	$y = 1$
r_2, r_1, u_2, u_1	leading to a final value of	$y = -1$.

For this program, the sequence of access events uniquely determines the final state of the computation.

While the access events themselves are constrained by the memory protection mechanism to form a linear sequence in which no two events coincide, the execution of the non-accessing part of the instructions will generally overlap in time. In fact, many different executions which greatly differ in the timing and overlaps of their non-accessing parts and instructions correspond to the same linear timing sequence of the accessing events, and hence yield the same final state. This proliferation of executions which all yield the same result and display essentially the same behavior makes the analysis of concurrent executions unnecessarily complicated.

Consequently, in order to reduce the complexity of analysis we use a simplified model in which the executions are restricted to be *interleaved.* An interleaved execution is one in which at any instant only one processor is executing an *elementary instruction* to its completion. Once the elementary instruction is completed, another processor may initiate an elementary instruction and proceed to complete it. Under this model, the execution proceeds as a sequence of discrete steps. In each step one enabled transition (instruction) is selected in one of the processes and is executed to completion.

The selection of the next process to be executed is personified by a *scheduler* who performs the selection. At each step of the computation the scheduler selects one process which has an enabled transition and lets that process execute one instruction (transition). For the sake of completeness we also allow the scheduler to arbitrarily insert an *idling* step in which no process is scheduled, no instruction is performed, and the values of all program and location variables remain the same. In the case that no enabled transition is available, an idling step is the only choice that the scheduler has thereafter. In such a case we say that the program is *deadlocked.* A special case of this situation is when the program has *terminated*, *i.e.*, all the processes have terminated.

When first encountered the model of *interleaved execution* may appear to be artificial and counterintuitive. In fact it seems to defeat the whole idea of concurrency – *concurrent (overlapping) execution* of instructions in different processes. Therefore we emphasize that the interleaving model is only a mathematical device for simplifying the analysis which proves to be adequate for the kind of non-quantitative analysis we consider here. That is, as long as we are not interested in questions about the timing of instructions and the running time of a program and make no assumptions about the relative speeds of the processors, the model of interleaved executions faithfully represents all the possible behaviors of the program.

We use the following definitions:

- An access to a variable in an instruction of a process P_i is defined to be *critical* if it is either a modification of a variable which is accessed by other processes or an access to a variable which is modifiable by other processes.

- An instruction is said to obey the *single (critical) access rule* if it contains at most one critical access.

We can then state the following result:

Proposition (single (critical) access): Interleaved executions of a program P, all of whose instructions obey the *single (critical) access rule*, faithfully represent all concurrent executions of P.

Thus, it is possible to represent by interleaving all possible situations arising under concurrency. Since this approach greatly simplifies the analysis, we will adopt this it in our treatment of concurrent programs.

One necessary exception to the single access rule is semaphores.

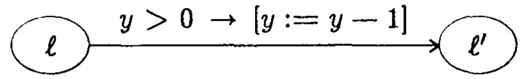
SEMAPHORES

Semaphores are devices for achieving synchronization in concurrent systems ([DIJ1]). They are special atomic instructions denoted by *request*(y) (also known as $P(y)$), and *release*(y) (also known as $V(y)$), operating on the *semaphore variable* y.

The *request* instruction

▸ ℓ : *request*(y)
ℓ' :

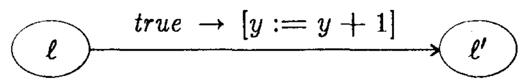
is equivalent to the single transition



The *release* instruction

▸ ℓ : *release*(y)
ℓ' :

is equivalent to the transition



Semaphores are considered atomic (primitive) even under concurrent execution. Therefore when programs are transformed to single access form, the semaphore instructions should be preserved as atomic and not broken up into single access instructions. No other operations can be performed on semaphore variables.

Usually the semaphore variable y is initialized to 1. A process reaching a *request*(y) instruction will proceed beyond it only if $y > 0$, and then it will decrement y by 1, setting it to 0. Thus a location containing a *request*(y) instruction can be used as a checkpoint, synchronizing the process with other processes containing *request*(y) and *release*(y) instructions operating on the same y.

Consider a concurrent program of form

$$y = 1$$

⋮	⋮		⋮
ℓ^1 : *request*(y)	ℓ^2 : *request*(y)		ℓ^k : *request*(y)
⋮	⋮		⋮
m^1 : *release*(y)	m^2 : *release*(y)		m^k : *release*(y)
⋮	⋮		⋮
— P_1 —	— P_2 —	• • •	— P_k —

Assume, for example, that P_1 arrived first at ℓ^1 when y was 1. It then went beyond ℓ^1 and set y to 0. As long as P_1 is between ℓ^1 and m^1, y will remain 0, and any other process, say P_2, which will attempt to go beyond its *request* statement ℓ^2 will be held there since the enabling condition $y > 0$ is false. It must wait there for y to turn positive, which can only be caused by P_1 performing the *release*(y) operation at m^1. Even if P_1 and P_2 reach ℓ^1 and ℓ^2 simultaneously, the atomicity of the *request* instruction (which is required for exactly this reason) ensures that only one process can gain access to its region lying between ℓ and m. This region is called a *critical section*, and our use of semaphores in this example ensures *mutual exclusion* of access to the critical sections; that is, at most one of the processes may execute its critical section at any instant. Semaphores may also be used for a variety of other signalling and synchronization tasks.

Mutual exclusion of critical sections is necessary whenever two or more processes need to access a shared variable or device (such as disk) and wish to be protected from interference or attempts by the other processes to access the same resource while doing so.

Example:

Consider once more Program BC_0 (Binomial Coefficient). In order to recast it in the single access form we notice that the variable y_3 is the critically shared variable. Hence, we have to break the instruction

$$\ell_1 : \quad y_3 := y_3 \cdot y_1$$

into the sequence

$$\begin{aligned} \ell_1 &: \quad t_1 := y_3 \cdot y_1 \\ \ell_1' &: \quad y_3 := t_1 \end{aligned}$$

Note that y_1 is modified only by P_1; hence its access at ℓ_1 is non-critical.

Similarly we have to break the instruction

$$m_3 : \quad y_3 := y_3/y_2$$

into

$$\begin{aligned} m_3 &: \quad t_2 := y_3/y_2 \\ m_3' &: \quad y_3 := t_2. \end{aligned}$$

Note that both the assignments $y_1 := y_1 - 1$, $y_2 := y_2 + 1$ and the test $y_1 + y_2 \leq n$ already satisfy the single access rule.

The problem now is that of interference between the two new processes. Consider for example an execution which includes the sequence:

$$\ell_1, \; m_3, \; \ell_1', \; m_3'.$$

Following this execution we find that while the instruction at ℓ_1' stores a certain value into y_3, it is immediately overwritten by the value stored into it by the instruction at m_3'. Thus the value of the computation performed in ℓ_1 is completely lost and the result is of course invalid. To prevent such a mishap we must protect each of the sequences (ℓ_1, ℓ_1') and (m_3, m_3') from interference by the other. The protection is done by using a semaphore variable y_4; the modified programs appears below:

Program BC (modified Binomial Coefficient):

$$y_1 := n, \quad y_2 := 0, \quad y_3 := 1, \quad y_4 := 1$$

ℓ_0 : *if* $y_1 = (n-k)$ *then go to* ℓ_e	m_0 : *if* $y_2 = k$ *then go to* m_e
ℓ_1 : *request*(y_4)	m_1 : $y_2 := y_2 + 1$
ℓ_2 : $t_1 := y_3 \cdot y_1$	m_2 : *loop until* $y_1 + y_2 \leq n$
ℓ_3 : $y_3 := t_1$	m_3 : *request*(y_4)
ℓ_4 : *release*(y_4)	m_4 : $t_2 := y_3/y_2$
ℓ_5 : $y_1 := y_1 - 1$	m_5 : $y_3 := t_2$
ℓ_6 : *go to* ℓ_0	m_6 : *release*(y_4)
ℓ_e : *halt*	m_7 : *go to* m_0
	m_e : *halt*

The mutually protected critical sections are (ℓ_2, ℓ_3, ℓ_4) and (m_4, m_5, m_6) respectively. Their exclusion ensures that each computed value of y_3 is assigned to y_3 without any interference. Under interleaved executions, BC computes the binomial coefficient and is in single reference form. ▮

Example:

Consider the following program CP modelling a consumer-producer situation:

Program CP (*Consumer Producer*) :

$$b := \Lambda, \quad s := 1, \quad cf := 0, \quad ce := N$$

— P_1 : Producer —	— P_2 : Consumer —
ℓ_0 : *compute* y_1	m_0 : *request*(cf)
ℓ_1 : *request*(ce)	m_1 : *request*(s)
ℓ_2 : *request*(s)	m_2 : $y_2 := head(b)$
ℓ_3 : $t_1 := b \circ y_1$	m_3 : $t_2 := tail(b)$
ℓ_4 : $b := t_1$	m_4 : $b := t_2$
ℓ_5 : *release*(s)	m_5 : *release*(s)
ℓ_6 : *release*(cf)	m_6 : *release*(ce)
ℓ_7 : *go to* ℓ_0	m_7 : *compute using* y_2
	m_8 : *go to* m_0

The program is in single access form. The producer P_1 computes a value into y_1 without using any other program variables; the computation details are irrelevant. It then adds y_1 to the end of the buffer b. The consumer P_2 removes the first element of the buffer into y_2 and then uses this value for its own purposes (at m_7). It is assumed that the maximal capacity of the buffer b is $N > 0$. The '*compute using* y_2' instruction references y_2 but does not modify any of the shared program variables.

In order to ensure the correct synchronization between the processes we use three semaphore variables:

- The variable s ensures that the accesses to the buffer are protected and provides exclusion between the sections (ℓ_3, ℓ_4, ℓ_5) and (m_2, m_3, m_4, m_5).
- The variable ce ("count of empties") counts the number of free available slots in the buffer b. It protects the buffer b from overflowing. The producer cannot deposit a value in the buffer if $ce = 0$, and when it does deposit a value, it decrements ce by 1. Since we start with $ce = N$, the producer cannot deposit more than N items before the consumer has removed any of them. The consumer, on the other hand, increments ce by 1 whenever it removes an item and creates a new vacancy.
- The variable cf ("count of fulls") counts how many items the buffer currently holds. It is initialized to 0, incremented by the producer whenever a new item is deposited, and decremented by the consumer whenever an item is removed. It ensures that the consumer does not attempt to remove an item from an empty buffer. ▮

FAIRNESS

Another problem with modelling concurrency by interleaving is *fairness*. Consider first a program with no semaphore instructions, and where the *full-exit condition* $E_\ell(\overline{y})$ at each nonterminal location ℓ (*i.e.*, $\ell \neq \ell_e$) is identically true, *i.e.*, $E_\ell(\overline{y}) = true$ for every $\overline{y}$. Note that the latter is true for every linear-text program without semaphores. Under these restrictions every process that has not yet terminated is enabled, *i.e.*, it always has an enabled transition, and if selected by the scheduler can always execute this transition. Running under true concurrency, every process will go on executing until it reaches its termination label ℓ_e.

In order to model the same property under interleaving execution we require the scheduler to be *fair*. By that we mean that no process which is ready to run (*i.e.*, enabled) will be neglected forever. Stated more precisely, we exclude infinite executions in which a certain process which has not terminated is never scheduled from a certain point on. Note that all finite terminating sequences are necessarily fair. This will also prevent the scheduler from going on an infinite spree of idling steps when at least one process is enabled.

Coming back to the more general situation which allows semaphore instructions, we have to consider the possibility that a nonterminated process is not continuously enabled. Furthermore, its being enabled may depend on the action of the other processes, since in general the full-exit condition $E_\ell(\overline{y})$ depends on the shared variables $\overline{y}$.

Our requirement of *fairness* for this more general case will be formulated as:

> We disallow infinite sequences in which a certain process is enabled infinitely often and is scheduled only a finite number of times.

Example:

Consider the simplest case of two processes synchronized by a semaphore:

$$y := 1$$

P_1	P_2
ℓ_0 : *request*(y)	m_0 : *request*(y)
ℓ_1 : *release*(y)	m_1 : *release*(y)
ℓ_2 : *go to* ℓ_0	m_2 : *go to* m_0
— P_1 —	— P_2 —

Obviously the infinite execution sequence (where we only mention the label arrived at as a result of the current transition)

$$\ell_1,\ \ell_2,\ \ell_0,\ m_1,\ m_2,\ m_0,\ \ell_1, \ell_2,\ \ell_0,\ m_1,\ m_2,\ m_0,\ \ldots$$

is fair. On the other hand the sequence:

$$\ell_1,\ \ell_2,\ \ell_0,\ \ell_1,\ \ell_2,\ \ell_0,\ \ldots$$

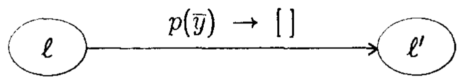
while constantly $\pi_2 = m_0$ is unfair. This is because whenever $\pi_1 = \ell_0$ or $\pi_1 = \ell_2$, P_2 is enabled. Thus in this sequence, even though P_2 is not continuously enabled (it is not enabled when $\pi_1 = \ell_1$), it is enabled infinitely often. Since P_2 is never scheduled while being enabled infinitely often this sequence is unfair.

In practice every scheduler which is fair satisfies a stronger requirement: it is fair within a finite bound, *i.e.*, no enabled process may be neglected for more than k instants of being enabled. Here k is a constant, characteristic of the scheduler.

Generalizing the semaphore instruction *request*(y) which waits for y to turn positive and then decrements it, we have the '*wait until* $p(\overline{y})$' and '*wait while* $p(\overline{y})$' instructions. They are modelled as follows:

▸ ℓ : *wait until* $p(\overline{y})$
ℓ' :

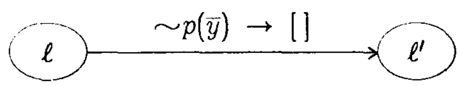
is represented by



and

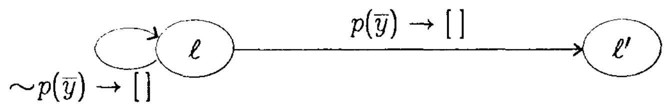
▸ ℓ : *wait while* $p(\overline{y})$
ℓ' :

is represented by



The *wait* instructions are similar to the *request* instruction in that the full-exit condition is not identically true. Thus for the '*wait until* $p(\overline{y})$' instruction, the full-exit condition $E_\ell(\overline{y})$ is equal to $p(\overline{y})$. Consequently fairness considerations ensure that if $p(\overline{y})$ turns true infinitely often while a process is waiting at ℓ it will eventually be scheduled (exactly when $p(\overline{y})$ is true) and proceed to ℓ'.

Let us compare the '*wait until* $p(\overline{y})$' instruction with the '*loop until* $p(\overline{y})$' instruction whose graph representation is



Note that the full-exit condition for this instruction is $E_\ell = true$. Thus even if $p(\overline{y})$ turns true infinitely often we are not assured of ultimately reaching ℓ'. This is so because the only requirement implied by fair scheduling is that if E_ℓ is infinitely often true the process waiting at ℓ must eventually be scheduled at an instant in which E_ℓ is true. However this instant may always happen to be one in which $p(\overline{y}) = false$ and the instruction executed is a transition back to ℓ.

The only condition that will guarantee for a *loop* instruction the eventual exit to ℓ' is that $p(\overline{y})$ becomes permanently true beyond a certain stage in the computation.

There are practical implications to the distinction between the *wait* and *loop* instructions. If we wish to implement an actual fair interleaving scheduler, it is easier to be fair to the *loop* instruction than to the *wait* instruction. Since for the *loop* instruction, E_ℓ is identically true, in order to be fair to a process which is at ℓ, the scheduler just has to make sure it does not neglect it and eventually comes around to scheduling it. In order to be fair to a *wait* instruction, whose full-exit condition is $p(\bar{y})$, we have to monitor the instants in which $p(\bar{y})$ is true. Then when it is observed that $p(\bar{y})$ is true many times the relevant process has to be eventually scheduled.

On the other hand, the use of a *wait* instruction implies greater efficiency since the scheduler may place the process executing a *wait* instruction on a suspension list, from which it will be removed only when $p(\bar{y})$ is true and the scheduler decides to schedule that process.

3. THE TEMPORAL DESCRIPTION OF PROGRAM PROPERTIES

As we have seen, the behavior of a concurrent program is characterized by the set of its fair execution sequences. We have also developed the formalism of temporal logic whose formulas are interpreted over sequences. We now combine the two and utilize temporal logic to state properties of the execution sequences of a given program, thus describing properties of the dynamic behavior of the program ([PNU1], [MP]).

In order to apply the general temporal formalism to execution sequences, it is necessary to introduce additional structure and special notation into the temporal language. For states we will consider "execution states" which each consist of the vector of current locations in the program and of the current values of all program variables at a certain stage in the execution. The accessibility relation between execution states will represent "derivability" by the program's execution. We will use predicates and propositions to describe properties of a single state, and modalities to describe properties of the execution leading from one state to another.

Consider a typical concurrent program

$$P \;=\; y := f_0(\overline{x});\; [P_1 \| \ldots \| P_m]$$

with input parameters $\overline{x} = (x_1, \ldots, x_k)$ and shared program variables $\overline{y} = (y_1, \ldots, y_n)$ over a domain D. (For simplicity, we do not consider many-sorted domains.)

An *execution state* for this program has the general structure

$$s = \langle \overline{\lambda}\, ; \overline{\eta} \rangle,$$

where

- $\overline{\lambda} = (\lambda_1, \ldots, \lambda_m)$ is the vector of current values held by the location variables $\overline{\pi}$. Thus $\lambda_i \in L_i$ is the label of the node in the transition graph of process P_i where execution is to resume next. (It is the label of the next instruction to be executed in the linear-text representation.)
- $\overline{\eta} = (\eta_1, \ldots, \eta_n) \in D^n$ is the vector of data values assumed by the program variables $\overline{y}$ in the state s. Thus $\eta_i \in D$ is the current value of y_i in s.

An *execution sequence* of a concurrent program is an infinite sequence of states:

$$\sigma \;=\; s_0,\, s_1,\, s_2,\, \ldots\, .$$

Corresponding to the structure of execution states and sequences we will consider temporal formulas with the following individual variables:

(a) *Local program variables:* $y_1, \ldots, y_n$.

These represent the current values of the program variables which of course may vary from one execution state to the other.

(b) *Local location variables:* $\pi_1, \ldots, \pi_m$.

These represent the location of each process in a given state. Each π_i will range over the set L_i.

(c) *Global variables*: $x_1, \ldots, x_k, u_1, u_2, \ldots$.

These are the input parameters $x_1, \ldots, x_k$, and auxiliary variables $u_1, u_2, \ldots$ which stay constant over the complete execution, *i.e.*, they do not vary from state to state. The auxiliary variables are used to express relations between local values in different states. For example:

$$\forall u[(y = u) \supset \Diamond(y = u + 1)]$$

expresses the statement that there will be a future instant in which the value of the variable y will be greater by 1 than its current value.

For a label $\ell \in L_j$, we abbreviate the atomic formula $\pi_j = \ell$ to $at\,\ell$, *i.e.*,

$$at\,\ell \text{ is true } \textit{iff } \pi_j = \ell,$$

which may therefore be considered a local proposition. Thus, for a given state $s = \langle \overline{\lambda}\,;\,\overline{\eta} \rangle$ and location $\ell \in L_j$, $at\,\ell$ is true at s if the process P_j is currently at ℓ, *i.e.*, $\lambda_j = \ell$.

More generally, for a set of labels $L \subseteq L_j$ the local proposition $at\,L$ is defined to be true if P_j is anywhere within L, *i.e.*,

$$at\,L \text{ is true } \textit{iff } \pi_j \in L.$$

If L consists of all the labels ℓ_i within a segment, *i.e.*, $L = \{\ell_a, \ell_{a+1}, \ldots, \ell_b\}$ for some $0 \leq a \leq b$, we will also write $at\,L$ as $at\,\ell_{a..b}$. Thus,

$$at\,\ell_{a..b} = at\{\ell_a, \ell_{a+1}, \ldots, \ell_b\} = \bigvee_{i=a}^{b} at\,\ell_i.$$

We proceed to give a precise definition for the set of *legal execution* sequences σ, corresponding to a given program P with input values $\overline{x} = \overline{\xi}$. There are three requirements which a legal execution sequence ought to fulfill:

A. Initialization

An execution sequence

$$\sigma = s_0, s_1, s_2 \ldots$$

is *properly initialized* if $s_0 = (\overline{\lambda_0}; \overline{\eta_0})$ has the structure:

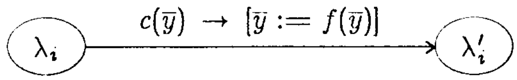
- $\overline{\lambda_0} = (\ell_0^1, \ldots, \ell_0^m)$, the set of initial locations in each of the processes;
- $\overline{\eta_0} = f(\overline{\xi})$, the initial values assumed by the program variables on initialization.

B. State to state transitions

An execution sequence σ is *admissible* if each $s_{k+1} = \langle \overline{\lambda}'\,;\,\overline{\eta}' \rangle$ is related to $s_k = \langle \overline{\lambda}\,;\,\overline{\eta} \rangle$ by one of the following rules:

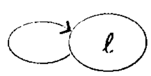
(a) *Idling step*: $s_{k+1} = s_k$ $(i.e., \overline{\lambda}' = \overline{\lambda}, \overline{\eta}' = \overline{\eta})$

(b) *An i-step*: For some i, $1 \leq i \leq m$, we have the following: The process P_i contains a transition



such that $\pi_i = \lambda_i$, $c(\overline{\eta}) = true$ (*i.e.*, the transition is enabled) and $\overline{\eta}' = f(\overline{\eta})$. For all $j, j \neq i$, we have $\lambda'_j = \lambda_j$.

Note that in the presence of self loops, *i.e.*,



we cannot always uniquely decide whether an idling step or a trivial i-step led from state s_k to state s_{k+1}.

C. Fairness or Justice

- An admissible sequence σ is *just* if there is no process P_i which is continuously enabled beyond a certain state s in the sequence σ, and only a finite number of steps of σ are i-steps.

Thus the notion of justice ensures that no process is indefinitely neglected. This notion is adequate for programs with no semaphore instructions.

- An admissible sequence σ is *fair* if there is no process P_i which is enabled an infinite number of times in σ, and only a finite number of steps of σ are i-steps.

Note that a fair sequence is also a just sequence. In addition to the assurances given by justice, fairness guarantees that no process will remain blocked at a semaphore instruction whose exit condition turns true infinitely often. For programs without semaphore instructions the notions of fairness and justice coincide. Consequently, our treatment will concentrate on fair executions.

Note that in checking for fairness we are allowed to take a given step both as an i-step and as a j-step if both interpretations are possible. Thus the following degenerate program

$$\ell_0 : \textit{ go to } \ell_0 \qquad\qquad m_0 : \textit{ go to } m_0$$

possesses the legal execution sequence

$$\langle(\ell_0, m_0); ()\rangle, \quad \langle(\ell_0, m_0); ()\rangle, \quad \ldots$$

Each step here may be interpreted as an idling step, a 1-step or a 2-step. Because of this possible multiple interpretation the sequence is indeed fair.

Consider the sequence corresponding to a terminating computation, *i.e.*, all processes have terminated. Since in a terminating state $\pi_i = \ell^i_e$ the process P_i is never enabled, the fairness criterion does not require further scheduling of P_i, and the only possible steps from that point on are idling steps. Thus our representation of a terminating computation as an infinite sequence in which from a certain point on all states are identical is consistent with fairness. This state, to which the sequence has "converged," is the terminal state.

- Every suffix of a properly $\bar{\xi}$-initialized, admissible, fair execution sequence is defined to be a $(P,\bar{\xi})$-*computation*. The set of all $(P,\bar{\xi})$-computations is denoted by $\mathcal{F}(P,\bar{\xi})$. By definition, this set is suffix closed, *i.e.*, if $\sigma \in \mathcal{F}(P,\bar{\xi})$, then $\sigma^{(i)} \in \mathcal{F}(P,\bar{\xi})$ for every $i \geq 0$.

For a given program P let $\varphi(\bar{x})$ be a restriction (precondition) on the input parameters $\bar{x}$. Usually φ characterizes the inputs we expect the program to operate on.

- A computation is said to be a (P,φ)-*computation* (*proper computation*) if it is a $(P,\bar{\xi})$-computation for some $\bar{\xi}$ such that $\varphi(\bar{\xi})$ is true.

- We define the set $\mathcal{F}(P,\varphi)$ to be the set of all (P,φ)-computations. Obviously $\mathcal{F}(P,\varphi)$ also has the suffix closure property.

- A formula w is $\mathcal{F}(P,\varphi)$-*valid* if it is true for every computation in $\mathcal{F}(P,\varphi)$. Such a formula is obviously an established valid property of all (P,φ)-computations. In the following sections we study the expression of program properties as $\mathcal{F}(P,\varphi)$-valid formulas.

Since most of our reasoning will be done in the context of a fixed program P and a fixed precondition φ, we introduce a special notation for $\mathcal{F}(P,\varphi)$ validity. We denote

$$\mathcal{F}(P,\varphi) \models w \quad \text{by} \quad \mathrel{|}\!\!\equiv w.$$

The statement $\mathrel{|}\!\!\equiv w$ thus means that w is true for every suffix of a fair, admissible execution of P which is initiated at $\overline{\ell_0} = (\ell_0^1, \ldots, \ell_0^m)$ with $\varphi(\bar{x})$ holding and $\bar{y} = f_0(\bar{x})$.

Facts of the form $\mathrel{|}\!\!\equiv w$ will serve as the basic statements in our specification and description of program properties. Consequently, we will discuss in later reports proof rules for deriving such statements.

The following is an important derivation:

$$\models w \quad \Rightarrow \quad \mathrel{|}\!\!\equiv w.$$

It states that if w is true for every possible sequence it is true in particular for every (P,φ)-computation. This enables us to transport all the generally known valid temporal statements ($\models$-valid) into reasoning about a particular program ($\mathrel{|}\!\!\equiv$-valid). Thus the following are $\mathrel{|}\!\!\equiv$-valid formulas:

$$\mathrel{|}\!\!\equiv \quad \Box\sim w \equiv \sim\Diamond w$$

$$\mathrel{|}\!\!\equiv \quad \Box(w_1 \supset w_2) \supset (\Box w_1 \supset \Box w_2)$$

$$\mathrel{|}\!\!\equiv \quad \Box(w \supset \bigcirc w) \supset (w \supset \Box w)$$

etc.

Another valid inference is

$$\mathrel{|}\!\!\equiv w \quad \Rightarrow \quad \mathrel{|}\!\!\equiv \Box w$$

This rule states that if w is true for all the (P,φ)-computations then $\Box w$ is also true for them. This rule is a direct consequence of the suffix closure property of $\mathcal{F}(P,\varphi)$. One can prove similarly that all the inference rules (numbers 56 to 66) proven in the earlier repertoire still hold after replacing $\models$ by $\models\!\!=$.

We will now review the expression of program properties by temporal formulas. The properties will be classified according to the form of the temporal formulas expressing them.

INVARIANCE (SAFETY) PROPERTIES

Consider first the class of program properties that hold continuously throughout all computations. They are expressible by formulas of the form:

$$\models \ \Box w.$$

Such a formula states that $\Box w$ holds for every computation, *i.e.*, w is an invariant of every computation. By the generalization rule this could have been written as $\models w$, but we prefer the above form since it emphasizes the invariant character of the properties in this class.

Note that the initial condition associated with the proper computation is:

$$at\,\overline{\ell_0} \ \wedge \ \overline{y} = f_0(\overline{x}) \ \wedge \ \varphi(\overline{x})$$

which characterizes the initial state for inputs $\overline{x}$ satisfying the precondition $\varphi(\overline{x})$. Here, $\overline{\ell_0} = (\ell_0^1, \ldots, \ell_0^m)$ is the set of initial locations in each of the processes. To emphasize the precondition $\varphi(\overline{x})$ we sometimes express $\models \Box w$ as

$$\models \ \varphi(\overline{x}) \supset \Box w.$$

A formula of this form therefore expresses an *invariance property*. The properties in this class are also known as *safety properties*, based on the premise that they ensure that "nothing bad will ever happen" ([LAM1]).

More generally, invariance properties can be expressed by formulas of the form

$$\models \ w_0 \ \supset \ \Box w.$$

This form may be used to state that a certain event implies the invariance of some other condition from that moment on. Under this interpretation w_0 is the triggering event whose occurrence causes the subsequent invariance of the property w.

We give below a sample of important properties falling under this category.

a. Partial Correctness

This property is meaningful only for programs in which each process contains a terminal location ℓ_e. We call such programs *terminating programs*, in contrast with *continuous* (or *cyclic programs*) whose proper behavior does not call for termination and therefore do not contain terminal locations.

Let $\varphi(\overline{x})$ be the precondition that restricts the set of inputs for which the program is supposed to be correct, and $\psi(\overline{x},\overline{y})$ the statement of its correctness, *i.e.*, the relation that should hold between the input values $\overline{x}$ and the output values $\overline{y}$. Then in order to state *partial correctness* with respect to a specification (φ,ψ) we can write:

$$\models \ \varphi(\overline{x}) \supset \Box\big(at\,\overline{\ell_e} \supset \psi(\overline{x},\overline{y})\big),$$

where $\overline{\ell_e} = (\ell_e^1, \ldots, \ell_e^m)$ is the vector of terminal locations in each of the processes. This formula claims that if the initial state satisfies the precondition, then in any state accessible from it: If that state happens to be an exit state, *i.e.* $\overline{\lambda} = \overline{\ell_e}$, then the relation $\psi(\overline{x},\overline{y})$ holds between the input parameters $\overline{x}$ and the current values of $\overline{y}$. Thus this formula states that all convergent φ-computations terminate in a state satisfying ψ, but it does not guarantee termination itself. Note that we rely on $\overline{x}$ being global and retaining its original value throughout the computation.

Example:

Let us consider as a concrete example, a single process program for computing $x!$ over the nonnegative integers.

Program F (Factorial Program):

$$
\begin{array}{ll}
 & y_1 := x, \ \ y_2 := 1 \\
\ell_0: & \textit{if } y_1 = 0 \textit{ then goto } \ell_e \\
\ell_1: & (y_1, y_2) := (y_1 - 1, y_1 \cdot y_2) \\
\ell_2: & \textit{goto } \ell_0 \\
\ell_e: & \textit{halt.}
\end{array}
$$

The statement of its partial correctness is

$$\models \ (x \geq 0) \supset \Box(at\,\ell_e \supset y_2 = x!),$$

where the initial condition associated with the proper computation is actually

$$at\,\ell_0 \ \wedge \ y_1 = x \ \wedge \ y_2 = 1 \ \wedge \ x \geq 0.$$

We are justified in regarding partial correctness as an invariance property since it is actually a part of a "network of invariants" normally used in the Invariant-Assertion Method; namely, for the Program F above:

$$
\begin{array}{lllll}
\models \ (x \geq 0) \ \supset \ \Box\{ & & [at\,\ell_0 & \supset & (y_1 \geq 0) \wedge (y_2 \cdot y_1! = x!)] \\
 & \wedge & [at\,\ell_1 & \supset & (y_1 > 0) \wedge (y_2 \cdot y_1! = x!)] \\
 & \wedge & [at\,\ell_2 & \supset & (y_1 \geq 0) \wedge (y_2 \cdot y_1! = x!)] \\
 & \wedge & [at\,\ell_e & \supset & (y_1 = 0) \wedge (y_2 = x!)] \ \}.
\end{array}
$$

And in fact, in order to prove the partial correctness property, we usually prove the invariance of this larger formula, from which partial correctness follows. ∎

Example:

As another example consider a program TN counting the number of nodes in a binary tree X.

Program TN (Counting the nodes of a tree):

$$S := (X),\quad C := 0$$

$$\begin{array}{ll} \ell_0: & \textit{if } S = (\,) \textit{ then goto } \ell_e \\ \ell_1: & (T, S) := (hd(S),\ tl(S)) \\ \ell_2: & \textit{if } T = \Lambda \textit{ then goto } \ell_0 \\ \ell_3: & C := C + 1 \\ \ell_4: & S := \ell(T) \cdot r(T) \cdot S \\ \ell_5: & \textit{goto } \ell_0 \\ \ell_e: & \textit{halt.} \end{array}$$

The program operates on a tree variable T and a variable S which is a stack of trees. The input variable X is a tree. The output is the value of the counter C. Each node in a tree may have zero, one or two descendants.

The available operations on trees are the functions $\ell(T)$ and $r(T)$ which yield the left and right subtrees of a tree T respectively. If the tree does not possess one of these subtrees the functions return the value Λ.

The stack S is initialized to contain the tree X. Taking the head and tail of a stack (functions hd and tl respectively) yields the top element and rest of the stack respectively. The operation in ℓ_1 pops the top of the stack into the variable T. The operation at ℓ_4 pushes both the right subtree and the left subtree of T onto the top of the stack.

At any iteration of the program, the stack S contains the list of subtrees of X whose nodes have not yet been counted. The iteration removes one such subtree from the stack. If it is the empty subtree, $T = \Lambda$, we proceed to examine the next subtree on the stack. If it is not the empty subtree we add one to the counter C and pushes the left and right subtrees of T to the stack. When the stack is empty, $S = (\,)$, the program halts.

Denoting by $|T|$ the number of nodes in a tree T we can express the statement of partial correctness of the program TN by:

$$\models\ \Box[at\,\ell_e \supset C = |X|].$$

The actual initial condition associated with the proper computation is

$$at\,\ell_0\ \wedge\ S = (X)\ \wedge\ C = 0.$$

Example:

As a more complex example consider again the program BC for the concurrent computation of a binomial coefficient.

The statement of partial correctness to be proved there is:

$$\models\ (0 \leq k \leq n)\ \supset\ \Box[(at\,\ell_e \wedge at\,m_e)\ \supset\ y_3 = \binom{n}{k}].$$

That is, every properly initialized execution of the program BC that terminates satisfies $y_3 = \binom{n}{k}$ at its termination point. The actual initial condition associated with the proper computation is

$$at\,\ell_0 \;\wedge\; at\,m_0 \;\wedge\; y_1 = n \;\wedge\; y_2 = 0 \;\wedge\; y_3 = 1 \;\wedge\; y_4 = 1 \;\wedge\; 0 \leq k \leq n. \quad \blacksquare$$

b. Clean Behavior

For every location in a program we can formulate a *cleanness* condition that states that the instruction at this location will execute successfully and will generate no execution faults (exceptions). Thus if the statement contains a division, the cleanness condition will include the clause specifying that the divisor is nonzero or not too small (to avoid arithmetic overflow). If the statement contains an array reference, the cleanness condition will state that the subscript expressions are within the declared range. Denoting the cleanness condition at location ℓ by α_ℓ, the statement of clean behavior is:

$$\models \;\; \varphi(\bar{x}) \;\supset\; \Box \bigwedge_{\ell} (at\,\ell \supset \alpha_\ell).$$

The conjunction is taken over all "potentially dangerous" locations in the program.

Example:

The factorial program F above should produce only natural number values during its computation. A cleanness condition at ℓ_1, which is clearly a critical point, is (under the precondition $x \geq 0$)

$$\models \;\; (x \geq 0) \;\supset\; \Box[at\,\ell_1 \supset (y_1 > 0)],$$

guaranteeing that the subtraction performed at ℓ_1 always yields a natural number. Note that we have not indicated that y_1 is an integer; such type considerations will be ignored in our discussions. $\blacksquare$

Example:

If a program contains the instruction

$$\ell: \quad \textit{if } y_1 > y_2 \textit{ then } y_1 := (S[i] \div y_2),$$

where $\div$ is the integer-division operator and the range of the array subscript i is between 1 and m, then the cleaness condition at ℓ can be expressed as follows:

$$\models \;\; \Box\{[at\,\ell \wedge (y_1 > y_2)] \;\supset\; [(1 \leq i \leq m) \wedge (y_2 \neq 0)]\}. \quad \blacksquare$$

Example:

A clean behavior statement for the tree node counting program TN is given by:

$$\models \;\; \Box[(at\,\ell_1 \supset S \neq ()) \;\wedge\; (at\,\ell_4 \supset T \neq \Lambda)].$$

This ensures that no attempt is made to pop an empty stack or to decompose an empty tree. ∎

Example:

In the binomial coefficient program BC an appropriate and crucial cleanness statement is given by:

$$\models \ (0 \leq k \leq n) \ \supset \ \Box\{at\, m_4 \ \supset \ [(y_2 \neq 0) \wedge (y_3 \, mod \, y_2 = 0)]\}.$$

That is, whenever we reach the location m_4 in a proper computation of BC, y_3 is evenly divisible by y_2. ∎

A general concern in the considerations of clean behavior is the compatibility of values with types. In the presence of dynamic types we should also worry about the compatibility of types.

c. Global and Local Invariants

Very frequently, invariant properties are not related to any particular location. In general, some properties may be invariant independent of the location. In these cases we speak of *global invariants*, *i.e.*, invariants unattached to any particular location. The expression of global invariance is even more straightforward. Thus, we write

$$\models \ \varphi(\overline{x}) \ \supset \ \Box\beta,$$

to state that property β holds at all times during a proper computation.

Example:

In the factorial program F above, to claim that y_1 is always a nonnegative integer, we may write:

$$\models \ (x \geq 0) \ \supset \ \Box(y_1 \geq 0).$$

Another valid global invariant for this program is:

$$\models \ (x \geq 0) \ \supset \ \Box(y_2 \cdot y_1! = x!),$$

which states that $y_2 \cdot y_1! = x!$ at all steps of the execution. ∎

Example:

For the binomial coefficient program BC, an appropriate global assertion would be:

$$\models \ (0 \leq k \leq n) \ \supset \ \Box[(n - k \leq y_1 \leq n) \ \wedge \ (0 \leq y_2 \leq k)]. \ ∎$$

Another interesting set of properties are invariance properties which are attached to particular locations, but not necessarily to the exit locations of the program. These properties are particularly important for programs which have no exits and are expected to run indefinitely.

We refer to such properties as *local invariants* and write

$$\models \ \Box(at\,\ell \supset \beta)$$

to indicate that a statement β is true whenever we are at a certain location ℓ. Partial correctness is actually a local invariant referring to the exit locations.

Example:

In the TN program for counting the nodes in a tree, we can express as a local invariant the fact which is true whenever we visit the location ℓ_0; namely,

$$\models \ \Box[at\,\ell_0 \ \supset \ (\sum_{t \in S}|t| + C = |X|)],$$

i.e., the sum of the number of nodes in all the subtrees currently in the stack plus the current value of the counter C is invariant at ℓ_0 and equals the number of nodes in the tree X. ∎

Invariants can also be used in the context of a program whose output is not necessarily apparent at the end of the execution; for example, a sequential program whose output is printed on an external file during the computation.

Example:

Consider the following program PR for printing the infinite sequence of successive prime numbers

2, 3, 5, 7, 11, 13, 17, ...

Program PR (Printing the prime numbers):

$$
\begin{array}{ll}
 & y_1 := 2 \\
\ell_0: & print(y_1) \\
\ell_1: & y_1 := y_1 + 1 \\
\ell_2: & y_2 := 2 \\
\ell_3: & if\ (y_2)^2 > y_1\ then\ goto\ \ell_0 \\
\ell_4: & if\ (y_1\, mod\, y_2) = 0\ then\ goto\ \ell_1 \\
\ell_5: & y_2 := y_2 + 1 \\
\ell_6: & goto\ \ell_3
\end{array}
$$

A part of the correctness statement for this program is:

$$\models \ \Box(at\,\ell_0 \supset prime(y_1));$$

it indicates that only primes are printed. ∎

Next we will examine some properties which are meaningful only for concurrent programs.

d. Mutual Exclusion

The notions of critical sections and mutual exclusion were introduced earlier, but let us briefly review them.

Consider two processes P_1 and P_2 being executed in parallel. Assume that each process contains a section $C_i \subseteq L_i$, for $i = 1, 2$, which includes some task critical to the cooperation of the two processes. For example, it might access a shared device (such as a disk) or a shared variable. If the nature of the task is such that it must never be done by both of them simultaneously, we call these sections *critical sections*. The property stating that the processes will never simultaneously execute their respective critical sections is called *mutual exclusion* with respect to this pair of critical sections.

The property of mutual exclusion for C_1 and C_2 can be described by:

$$\models \ \varphi(\bar{x}) \ \supset \ \Box \sim (at\, C_1 \wedge at\, C_2).$$

This states that it is never the case that the joint execution of the processes reaches C_1 and C_2 simultaneously.

Example:

Consider again the consumer-producer program CP. The sections

$$C_1 = \{\ell_3, \ell_4, \ell_5\} \quad \text{in } P_1$$

and

$$C_2 = \{m_2, m_3, m_4, m_5\} \quad \text{in } P_2$$

are obviously critical sections since they make several accesses to the shared variable b. In order to obtain the correct result it must be ensured that no other accesses to b are made during the computation involving b.

The mutual exclusion property in this case can be expressed by:

$$\models \ \Box \sim (at\, C_1 \wedge at\, C_2),$$

where the initial condition associated with the proper computations is:

$$at\,\ell_0 \ \wedge \ at\, m_0 \ \wedge \ b = \Lambda \ \wedge \ s = 1 \ \wedge \ cf = 0 \ \wedge \ ce = N.$$

The formula states that we can never simultaneously be in both critical sections C_1 and C_2. Note that actually it suffices to prove

$$\models \ \Box \sim (at\,\ell_3 \wedge at\, m_2).$$

This is so because there exists an execution in which $at\,\ell_3 \wedge at\, m_2$ in some state if and only if there exists an execution in which $at\, C_1 \wedge at\, C_2$ in some state. ∎

Example:

Similarly a statement of mutual exclusion for the program BC computing the binomial coefficient is given by:

$$\models \ (0 \leq k \leq n) \supset \square\sim(at\,\ell_{2..4} \wedge at\,m_{4..6}).$$

Here, we follow our convention,

$$at\,\ell_{2..4} \ \text{ denotes } \ \pi_1 \in \{\ell_2, \ell_3, \ell_4\}$$

and

$$at\,m_{4..6} \ \text{ denotes } \ \pi_2 \in \{m_4, m_5, m_6\}. \ \blacksquare$$

e. Deadlock Freedom

A concurrent program consisting of m processes is said to be *deadlocked* if no process is enabled. This leaves the idling step as the only possible choice of the scheduler. The rest of the computation will therefore consist of an endless repetition of the current deadlocked state. Clearly in a deadlock situation each process P_j must be blocked at a location $\ell \in L_j$ whose full-exit condition E_ℓ is false for the current value $\overline{\eta}$ of $\overline{y}$. Therefore the only potential deadlock locations are those ℓ for which E_ℓ is not identically true. We refer to such locations as *waiting locations.* The terminal location ℓ_e is also considered to be a waiting location. However, the special case in which *all* processes are at their respective ℓ_e locations is not considered to be a deadlock but rather a *termination.*

Let us therefore consider a tuple $\overline{\ell} = (\ell^1, \ldots, \ell^m)$ of waiting locations, $\ell^j \in L_j$, not all of which are terminal locations. Let $E_1, \ldots, E_m$ be their associated full-exit conditions. To prevent a deadlock at $\overline{\ell}$ we require:

$$\models \ \varphi(\overline{x}) \ \supset \ \square\Big(\bigwedge_{j=1}^{m} at\,\ell^j \supset \bigvee_{j=1}^{m} E_j(\overline{y})\Big).$$

This indicates that whenever all the processes are each at ℓ^j, $j = 1, \ldots, m$, at least one of them is enabled. The corresponding process can then proceed and deadlock is averted.

In order to eliminate the possibility of a deadlock in the full program, we must impose a similar requirement for every possible n-tuple of waiting locations, excluding $\overline{\ell_e} = (\ell_e^1, \ldots, \ell_e^m)$.

Example:

In the consumer producer program CP, the complete deadlock freedom condition will be expressed as

$$\begin{array}{lll} \models \ \square\{ & & [(at\,\ell_1 \wedge at\,m_0) \supset (ce > 0 \vee cf > 0)] \\ & \wedge & [(at\,\ell_1 \wedge at\,m_1) \supset (ce > 0 \vee s > 0)] \\ & \wedge & [(at\,\ell_2 \wedge at\,m_0) \supset (s > 0 \vee cf > 0)] \end{array}$$

$$\wedge \quad [(at\,\ell_2 \wedge at\,m_1) \supset (s > 0)] \}. \quad ∎$$

f. generalized deadlock

We may generalize the definition of waiting locations to also include looping instructions of the form:

$$\ell: \quad \textit{loop until } p(\overline{y}) \qquad \text{or} \qquad \ell: \quad \textit{loop while } \sim p(\overline{y}).$$

Obviously, being trapped at a tuple $(\ell^1, \ldots, \ell^m)$ some of whose locations are looping locations, with $\overline{y} = \overline{\eta}$ such that $p(\overline{\eta}) = \textit{false}$ for their escape conditions, is just as bad as a deadlock. Formally such a situation is not a deadlock since the execution of the self-transitions in the looping locations is not officially an idling step. But it is also self-evident that these steps cannot alter the state and the computation will remain trapped forever.

Let us therefore call a *generalized deadlock* situation to be a state $s = (\ell^1, \ldots, \ell^m; \overline{\eta})$ such that each ℓ^i is either a waiting location or a looping location, and such that $\mathcal{E}_i(\overline{\eta}) = \textit{false}$ for each $i = 1, \ldots, m$. The *escape condition* $\mathcal{E}_i(\overline{y})$ corresponding to location ℓ^i is taken as the exit condition $E_{\ell^i}(\overline{y})$ if ℓ^i is a semaphore location, *false* if ℓ^i is a terminal location ℓ^i_e, and the condition for getting out of the self-loop if ℓ^i is a looping instruction of the form

$$\ell^i: \quad \textit{loop until } \mathcal{E}_i(\overline{y}) \qquad \text{or} \qquad \ell^i: \quad \textit{loop while } \sim \mathcal{E}_i(\overline{y}).$$

Then again the statement ensuring prevention of generalized deadlock at a tuple $\overline{\ell} = (\ell^1, \ldots, \ell^m)$ is the requirement

$$\models \Box\Big(\bigwedge_{j=1}^{m} at\,\ell^j \quad \supset \quad \bigvee_{j=1}^{m} \mathcal{E}_j(\overline{y})\Big).$$

Example:

Consider the binomial coefficient program BC. A statement of the impossibility of general deadlock at the potentially dangerous locations is given by:

$$\begin{aligned} \models \quad (0 \leq k \leq n) \supset \Box\{ \quad & [(at\,\ell_1 \wedge at\,m_3) \supset (y_4 > 0)] \\ \wedge \quad & [(at\,\ell_1 \wedge at\,m_e) \supset (y_4 > 0)] \\ \wedge \quad & [(at\,\ell_e \wedge at\,m_2) \supset (y_1 + y_2 \leq n)] \\ \wedge \quad & [(at\,\ell_e \wedge at\,m_3) \supset (y_4 > 0)] \\ \wedge \quad & [(at\,\ell_1 \wedge at\,m_2) \supset (y_4 > 0 \vee y_1 + y_2 \leq n)]\}. \end{aligned}$$

This statement ensures that if execution is at (ℓ_1, m_3) then $y_4 > 0$ and one of the processes is able to proceed; if one of the processes is ever at its terminal location the other process is not deadlocked at its *request* instruction or trapped at its *loop* instruction; and if the execution is ever at (ℓ_1, m_2) then either $y_4 > 0$ or $y_1 + y_2 \leq n$, thus either enabling P_1 or permitting P_2 to exit from its self-loop. ∎

EVENTUALITY (LIVENESS) PROPERTIES

A second category of properties are those expressible by formulas of the form:

$$\models \; w_1 \supset \Diamond\, w_2.$$

This formula states that for every proper computation, if w_1 is initially true then w_2 must eventually be realized. In comparison with invariance properties that only describe the preservation of a desired property from one step to the next, an eventuality property guarantees that some event will finally be accomplished. It is therefore more appropriate for the statement of goals which may need many steps to be realized.

Note that because of the suffix closure of the set of proper computations this formula is equivalent to:

$$\models \; \Box(w_1 \supset \Diamond\, w_2)$$

which states that whenever w_1 arises during the computation it will eventually be followed by the realization of w_2.

A property expressible by such a formula is called an *eventuality (liveness) property* ([OL]). Following are some samples of eventuality properties.

a. Total Correctness

This property, like partial correctness, is meaningful only for programs with terminal locations, *i.e.*, programs that are expected to terminate in contrast to continuous (cyclic) programs.

A program is said to be *totally correct* with respect to a specification (φ, ψ), if for all input values $\bar{x}$ satisfying $\varphi(\bar{x})$, termination is guaranteed, and the output values $\bar{y}$ upon termination satisfy $\psi(\bar{x}, \bar{y})$. Once more, let $\overline{\ell_e}$ denote the exit points of the program. Total correctness w.r.t. (φ, ψ) is expressible by:

$$\models \; \varphi(\bar{x}) \;\supset\; \Diamond(at\,\overline{\ell_e} \;\wedge\; \psi(\bar{x}, \bar{y})).$$

This says that if we have an admissible execution sequence beginning in a state which is at locations $\overline{\ell_0}$ and has values $\bar{y} = f_0(\bar{x})$ where $\varphi(\bar{x})$ is true, then later in that execution sequence we are guaranteed to have a state which is at $\overline{\ell_e}$ and satisfies $\psi(\bar{x}, \bar{y})$.

Example:

The statement of total correctness for the factorial program F is:

$$\models \; (x \geq 0) \supset \Diamond(at\,\ell_e \;\wedge\; y_2 = x!). \quad \blacksquare$$

Example:

The expression of total correctness for the tree node counting program TN is given by:

$$\models \; \Diamond(at\,\ell_e \;\wedge\; C = |X|). \quad \blacksquare$$

Example:

The statement of total correctness for the binomial coefficient program BC is given by:

$$\models\ (0 \leq k \leq n)\ \supset\ \Diamond[at\,\ell_e\ \wedge\ at\,m_e\ \wedge\ y_3 = \tbinom{n}{k}].\quad \blacksquare$$

b. Intermittent Assertions

Eventuality formulas enable us to express a causality relation between any two events, not only between program initialization and termination but also between events arising during the execution. This becomes especially important when discussing *continuous (cyclic) programs, i.e.*, programs that are not supposed to terminate but are to operate continuously. The general form of such an eventuality is:

$$\models\ (at\,\ell \wedge \phi) \supset \Diamond(at\,\ell' \wedge \phi')$$

and it claims that whenever (in a proper computation) ϕ arises at ℓ we are guaranteed of eventually reaching ℓ' with ϕ' true. This is the exact formalization of the basic *Intermittent-Assertion statement* ([BUR], [MW]):

"If sometime ϕ at ℓ then sometime ϕ' at ℓ'."

Example:

Consider the program TN for counting the number of nodes in a tree. An important intermittent assertion that serves as a basis for the proof of its correctness is:

$$\models\ [at\,\ell_0\ \wedge\ S = u \cdot s\ \wedge\ C = c]\ \supset\ \Diamond[at\,\ell_0\ \wedge\ S = s\ \wedge\ C = c + |u|].$$

Here, u, s and c are used in the role of *global* variables, while S and C are local program variables. This statement says that being at ℓ_0 with a nonempty stack ensures a later arrival to ℓ_0. In a subsequent arrival (not necessarily the next one), the top element of the stack will be removed and the value of C will have been incremented by the number of nodes in the top element.

Example:

Consider again the program PR for printing successive prime numbers. Under the invariance properties we expressed the claim that *nothing but primes is printed*

(1) $\models\ \Box(at\,\ell_0 \supset prime(y_1)).$

Now we can state that the proper sequence of primes is produced. The property that *every prime number is printed* can be expressed by

(2) $\models\ [at\,\ell_0\ \wedge\ y_1 = 2\ \wedge\ prime(u)]\ \supset\ \Diamond(at\,\ell_0\ \wedge\ y_1 = u).$

In conjunction with the invariance property (1), this statement guarantees that all printed results are primes

$$2, \quad 3, \quad 5, \quad 7, \quad 11, \quad 13, \quad 17, \quad \ldots,$$

but they do not guarantee that some primes are not printed more than once or out of sequence. For example, the sequence of integers

$$\begin{array}{ccccccccccc} 3, & 2, & 5, & 3, & 7, & 5, & 11, & 7, & 13, & 11, & \ldots \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \end{array}$$

satisfies the statements above.

We thus have to add an additional statement that will guarantee that the printed sequence is exactly the desired one. We have to be careful in devising a solution: Note that the statement

$$[at\,\ell_0 \wedge y_1 = u] \supset \square(at\,\ell_0 \supset y_1 > u)$$

does not resolve the problem! Why?

The property that *the primes are printed in order* can be expressed by

(3) $$\models \; [at\,\ell_1 \wedge y_1 = u] \supset \square(at\,\ell_0 \supset y_1 > u).$$

This ensures monotonicity for any future visit to ℓ_0. ∎

The following properties are of interest mainly for concurrent programs having more than one process.

c. Accessibility

Consider again a process that has a critical section C. In the previous discussion we have shown how to state exclusion (or protection) for that section. A related and complementary property is *accessibility*. That is, if a process wishes to enter its critical section it will eventually get there and will not be indefinitely held up by the protection mechanism. Obviously a foolproof protection mechanism is worthless if it does not eventually admit the process into its critical section.

Let ℓ_1 be a location just before the critical section. The fact that the process is at ℓ_1 indicates an intention to enter the critical section. Let C be the set of locations in the critical section. The property of accessibility can then be expressed by

$$\models \; at\,\ell_1 \supset \diamond\, at\,C;$$

namely, whenever the program is at ℓ_1, it will eventually get into C.

A correct construction of critical sections should ensure these two complementary properties: protection (exclusiveness) and accessibility.

Example:

For the consumer-producer program CP, we wish to express the property that whenever the producer is at ℓ_1 it will eventually get to ℓ_3 and be able to deposit y_1 in the buffer. A symmetric statement expresses accessibility for the consumer: whenever the consumer is at m_0 it will eventually get to m_2. The conjunction of these two properties, expressing the accessibility property of the program, is given by:

$$\models \; [at\,\ell_1 \supset \Diamond\, at\,\ell_3] \;\; \wedge \;\; [at\,m_0 \supset \Diamond\, at\,m_2].$$

d. Liveness

A more general class of eventuality properties arises when we consider the notion that *the computation of any particular process must eventually progress*. Here we do not necessarily restrict ourselves to locations containing semaphore instructions.

Consider an arbitrary non-terminal location ℓ in some process P_i, *i.e.*, $\ell \neq \ell_e$ for that process. If the computation of this process is to proceed we cannot remain blocked at ℓ due to a failure of the scheduler to schedule process P_i. Assuming that our program contains self-loops only for waiting purposes, such as in the *loop* instruction, progress in P_i is observable by seeing P_i moving from a state of $at\,\ell$ to a state of $\sim at\,\ell$. Consequently, the property of *liveness* for a general location ℓ, $\ell \neq \ell_e$, can be expressed by:

$$\models \; at\,\ell \supset \Diamond \sim at\,\ell,$$

i.e., if we arrive at this location we will eventually move out. In fact we can simplify this formula to

$$\models \; \Diamond \sim at\,\ell$$

which is equivalent to

$$\models \; \sim \Box\, at\,\ell,$$

meaning that we cannot get blocked at the location ℓ.

The property of liveness is also known as *absence of livelock* or *freedom from individual starvation*. A *livelock* (or *individual starvation*) is defined as a situation in which *some* processes which are not in a terminal location cannot proceed even though the full program may still progress by having some other processes execute. Note that this is a stronger requirement than the absence of a (generalized) deadlock. As long as at least one of the processes can proceed the program is not deadlocked.

e. Responsiveness

A very important class of programs that are usually modeled as concurrent programs are operating systems and real-time programs such as airline reservation systems and other online data-base systems. These programs can conveniently be considered as *continuous* (*cyclic*) programs which are to run forever. A halt in these programs usually indicates an error condition. Consequently these programs are not run for their end results but for the effects produced during their endless operation. Thus the notions of total and partial correctness are meaningless and have to be replaced by statements about the programs' continuous behavior.

A property usually expected of such programs is *responsiveness.*

Example:

Consider a continuous program (*granter*) G modelling an operating system. Assume that it serves a number of customer programs (*requesters*) $R_1, \ldots, R_t$ by scheduling a shared resource between them. The resource here can be a shared disk, main memory, etc. Let the customer programs communicate with the operating system concerning the resource via a set of boolean variables $\{r_i, g_i\}$, for $i = 1, \ldots, t$. Here, r_i is set to *true* by the customer program R_i to signal a *request* for the resource; g_i is set to *true* by G signalling to R_i that it has been *granted (allocated)* the resource. After using the resource, the customer R_i *releases* the resource back to the system G by setting r_i to *false.* This release is then *acknowledged* by the system G by setting g_i to *false.*

To summarize:

R_i signals a request $\Rightarrow$ $r_i := \mathit{true}$

G allocates a resource $\Rightarrow$ $g_i := \mathit{true}$

R_i releases the resource $\Rightarrow$ $r_i := \mathit{false}$

G acknowledges the release $\Rightarrow$ $g_i := \mathit{false}.$

The statement that the operating system fairly responds to the customer requests – *responsiveness* – is given by:

$$a_i : \quad r_i \supset \Diamond g_i,$$

i.e., whenever r_i becomes *true,* eventually g_i will turn *true.* Note that this statement does not stipulate that r_i becomes true when G is at a particular location. Consequently it can express events such as interrupts or unsolicited signals which may occur at any arbitrary moment.

Similarly we have to ensure that the system acknowledges the release of the resource by turning g_i to *false*:

$$b_i : \quad \sim r_i \supset \Diamond \sim g_i.$$

Furthermore, the system cannot hope to operate successfully if it does not enjoy the cooperation of the customer programs. For example, the system cannot promise R_2 an eventual grant of the resource if R_1, who currently holds the resource, does not ever intend to release it. Consequently we will expect the R_i's to satisfy some proper behavior requirements, namely for each i:

$$c_i : \quad g_i \supset \Diamond \sim r_i.$$

This statement ensures that when the resource is granted to R_i, it will eventually be released.

To these statements we will usually add some invariance statements ensuring the correct continuous behavior of G. One such statement is

$$d: \quad \Box(\sum_{i=1}^{t} g_i \leq 1)$$

meaning that at any particular time the system grants the resource to at most one requester. This is a type of a mutual exclusion.

Denote the correct behavior statement of G by

$$\psi \quad = \quad \bigwedge_{i=1}^{t} a_i \ \wedge \ \bigwedge_{i=1}^{t} b_i \ \wedge \ d$$

and the correct behavior expected from the R_i's by

$$\varphi \quad = \quad \bigwedge_{i=1}^{t} c_i$$

The problem of proving the correct behavior of G can be approached in two different ways:

- Consider a concurrent program P that consists of G alone. The r_i's and g_i's are then considered as input/output variables, where the r_i's are supposed to be set by the external agents $R_1, \ldots, R_t$.

  For this program we would prove:

  $$\models \ \Box\varphi \supset \Box\psi.$$

  That is, provided the external communication φ continuously behaves properly we can promise the correct behavior ψ of G.

- As another alternative consider the concurrent program P that consists of G running together with $R_1, \ldots, R_t$, *i.e.*

  $$P \ = \ (\bar{r}, \bar{g}) := (\textit{false}, \ldots, \textit{false}); \ [G \| R_1 \| \ldots \| R_t].$$

  For each R_i here we substitute a simplified model that guarantees to maintain $\Box c_i$. Such a model can be represented as:

  ℓ_0 : *execute*
  ℓ_1 : r_i := *true*
  ℓ_2 : *wait until* g_i
  ℓ_3 : *compute* {*use resource*}
  ℓ_4 : r_i := *false*
  ℓ_5 : *wait until* $\sim g_i$

ℓ_6 : *go to* ℓ_0

— Customer Program R_i —

If we believe that our model for R_i faithfully represents the real R_i as far as communication with G is concerned, we can proceed to prove

$$\models \; \Box(\varphi \land \psi)$$

to ensure the correct behavior of P.

Thus the two modelling alternatives available to us are the following: either considering G alone communicating with the external world via the r_i, g_i variables, or considering a combined system of G together with $R_1, \ldots, R_t$. In the first case the proper behavior of the external world has to be promised through a continuous maintainance of φ. In the second case the proper behavior of the R_i's is proven at the same time as the proper behavior of G.

The same analysis can of course be conducted for other situations where a program communicates with external devices and is expected to respond properly to incoming signals. ∎

The application of the temporal formalism to the problems of responsiveness points out its power. Invariances and total correctness are long-known properties and many special formal systems and methodologies have been proposed and successfully implemented for their analysis and proofs. The temporal logic contribution to this problem is a uniform treatment and an explicit direct expressibility. In contrast, the discussion of responsiveness is relatively recent; no prior formalism addressed itself to the description and proof of these properties.

PRECEDENCE (UNTIL) PROPERTIES

The third class of properties to be considered are those properties which are expressible using the *until* operator.

In their simplest form they will be expressed by statements of the type:

$$\models \; w_1 \,\mathcal{U}\, w_2.$$

This statement says that in all proper computations of P there will be a future instance in which w_2 holds and such that w_1 will hold until that instance. Recall that the formal meaning of the until operator was given by

$$w_1 \,\mathcal{U}\, w_2\big|_\sigma^\alpha = \mathit{true} \quad \mathit{iff} \quad \text{for some } k \geq 0, \begin{bmatrix} w_2\big|_{\sigma^{(k)}}^\alpha = \mathit{true} \text{ and} \\ \text{for all } i,\ 0 \leq i < k,\ w_1\big|_{\sigma^{(i)}}^\alpha = \mathit{true}. \end{bmatrix}$$

Note that we require $i < k$ and not $i \leq k$. Thus, the formula $w_1 \,\mathcal{U}\, w_2$ expresses the *exclusive* form of the until operator since w_1 is required to hold *until* the instant that w_2 becomes true but not including that instant. The corresponding *inclusive* until property that requires w_1 to be true up to and including the instant in which w_2 becomes true can be expressed by the formula

$$w_1 \,\mathcal{U}\, (w_1 \land w_2).$$

The *until* operator is also very useful in expressing precedence relations between events. We define the derived *precede* operator $\mathcal{P}$ by:

$$w_1 \,\mathcal{P}\, w_2 \quad \text{is} \quad \sim((\sim w_1)\,\mathcal{U}\, w_2).$$

This makes $\mathcal{P}$ the dual of $\mathcal{U}$ in a similar way to $\Box$ being the dual of $\Diamond$. The statement $w_1 \,\mathcal{P}\, w_2$, read w_1 *precedes* w_2, states that if w_2 ever happens it will not happen until w_1 happens first. This is equivalent to stating that the first instance of w_1 (observed from the present) strictly precedes the first instance of w_2. The formal meaning of the precede operator can be given by

$$w_1 \,\mathcal{P}\, w_2\big|_{\sigma}^{\alpha} = \mathit{true} \quad \mathit{iff} \quad \text{for every } k \geq 0, \begin{bmatrix} \text{if } w_2\big|_{\sigma^{(k)}}^{\alpha} = \mathit{true} \\ \text{then for some } i,\ 0 \leq i < k,\ w_1\big|_{\sigma^{(i)}}^{\alpha} = \mathit{true}. \end{bmatrix}$$

Note that we have again $i < k$ and not $i \leq k$. Thus, the precedes operator $\mathcal{P}$ is again an *exclusive* operator, expressing strict precedence between w_1 and w_2.

If we wish to express *inclusive* precedence, allowing the first instances of w_1 and w_2 to coincide, we may use

$$w_1 \,\mathcal{P}\, (\sim w_1 \wedge w_2).$$

To show that this indeed expresses inclusive precedence, we may substitute $\sim w_1 \wedge w_2$ for w_2 in the definition above to obtain after some manipulation:

$$w_1 \,\mathcal{P}\, (\sim w_1 \wedge w_2)\big|_{\sigma^{(k)}}^{\alpha} = \mathit{true} \quad \textit{if and only if} \quad \begin{bmatrix} \text{for every } k \geq 0, \\ \text{if } w_2\big|_{\sigma^{(k)}}^{\alpha} = \mathit{true} \\ \text{then } w_1\big|_{\sigma^{(k)}}^{\alpha} = \mathit{true} \\ \qquad \text{or for some } i,\ 0 \leq i < k,\ w_1\big|_{\sigma^{(i)}}^{\alpha} = \mathit{true} \end{bmatrix}$$

showing that the first instance of w_2 either coincides with an instance of w_1 or is preceded by such an instance.

While $w_1 \,\mathcal{U}\, w_2$ implies that w_2 is bound to happen, this is not guaranteed by $w_1 \,\mathcal{P}\, w_2$. In fact, if w_2 never happens then $w_1 \,\mathcal{P}\, w_2$ holds for every w_1.

Several obvious properties of the *precedes* operator may be derived from corresponding properties of the $\mathcal{U}$ operator and the definition of $\mathcal{P}$. Among them are:

1. $\models \; w \,\mathcal{P}\, w \;\equiv\; \Box{\sim}w$
2. $\models \; w_1 \,\mathcal{P}\, w_2 \;\wedge\; w_2 \,\mathcal{P}\, w_3 \;\supset\; w_1 \,\mathcal{P}\, w_3$
3. $\models \; w_1 \,\mathcal{P}\, w_2 \;\equiv\; \sim w_2 \wedge [w_1 \vee \bigcirc(w_1 \,\mathcal{P}\, w_2)]$
4. $\models \; \Box{\sim}w_2 \;\supset\; w_1 \,\mathcal{P}\, w_2$
5. $\models \; w_1 \,\mathcal{P}\, w_2 \;\vee\; w_2 \,\mathcal{P}\, w_1 \;\vee\; \Diamond(w_1 \wedge w_2)$
6. $\models \; w_1 \,\mathcal{P}\, w_2 \;\vee\; w_2 \,\mathcal{P}\, (\sim w_2 \wedge w_1)$
7. $\models \; w_1 \,\mathcal{U}\, w_2 \;\equiv\; \sim(\sim w_1 \,\mathcal{P}\, w_2).$

Statement 1 says that w may precede itself *iff* it never happens, since no event can come before the first occurrence of that event.

Statement 2 indicates the transitivity of the precedence relation. It says that if w_1 precedes w_2 which precedes w_3 then w_1 precedes w_3.

Statement 3 gives an inductive characterization of the $\mathcal{P}$ operator. It says that w_1 precedes w_2 *iff* w_2 is presently false and either w_1 is true now or w_1 precedes w_2 when observed from the next instant.

Statement 4 says that if w_2 never happens then obviously w_1 precedes w_2, for every w_1.

Statements 5 and 6, each characterizes the linearity of time. Statement 5 says that for every two events w_1 and w_2, either w_1 precedes w_2 or w_2 precedes w_1 or both occur at the same time. Statement 6 says that for every two events w_1 and w_2, either w_1 strictly precedes w_2 or w_2 weakly precedes w_1.

Statement 7 shows that the $\mathcal{U}$ operator itself is expressible by the $\mathcal{P}$ operator.

We will consider formulas involving the $\mathcal{P}$ operator as belonging to the class of *until* properties. We discuss below several subclasses of properties involving the $\mathcal{U}$ and $\mathcal{P}$ operators.

a. Safe Liveness

We may interpret invariance properties as an assurance that nothing bad will happen, and liveness properties as a promise that something good will eventually happen. Consistent with this, we may want to ascertain that nothing bad happens until something good happens. This is exactly expressible by

$$\models \quad w_1 \, \mathcal{U} \, w_2,$$

where w_1 is a safety property that we wish to maintain (*e.g.*, clean behaviour and global assertions), while w_2 is a liveness property that we want ultimately to achieve (*e.g.*, termination and correctness). It is recommended that a full specification of a program should always be expressed as an until expression $\models w_1 \, \mathcal{U} \, w_2$, *i.e.*, achieve w_2 while maintaining w_1.

In some cases the "until" notation is just a conveniently expressed combination of safety and liveness properties since:

$$\models \quad (\Box w_1 \wedge \Diamond w_2) \supset w_1 \, \mathcal{U} \, w_2.$$

However the more interesting case is when w_1 holds up to but not including the instant in which w_2 happens. Then it is no longer true that $\Box w_1$ is a program-valid statement.

The *until* operator can also be used to express "first-time" properties. Recall that a formula of form

$$\models \quad (at\,\ell \wedge \phi) \supset \Diamond(at\,\ell' \wedge \phi')$$

expresses the *some-time* property: If the program is at ℓ and ϕ is true, then sometime (eventually) the program must reach ℓ' with ϕ' being true. Similarly, a formula of form

$$\models \quad (at\,\ell \wedge \phi) \supset [(\sim at\,\ell') \, \mathcal{U} \, (at\,\ell' \wedge \phi')]$$

expresses the *first-time* property: If the program is at ℓ and ϕ is true, then sometime the program must reach ℓ', and on the *first visit*, ϕ' will be true.

Example:

The safety and liveness properties for the binomial coefficient program BC can be stated as:

$$\begin{aligned}\models\ (0 \le k \le n)\ \supset\ & \\ \{\ [(at\, m_4\ \supset\ (y_2 \ne 0)\ \ \wedge\ \ (y_3\, mod\, y_2 = 0))& \\ \wedge\,(n-k \le y_1 \le n)\ \ \wedge\ \ (0 \le y_2 \le k)]& \\ \mathcal{U}& \\ [at\,\ell_e \wedge at\,m_e \wedge y_3 = \tbinom{n}{k}]\ \}.&\end{aligned}$$

That is, achieve termination and correct result while maintaining clean behavior and global invariances. ∎

b. Absence of Unsolicited Response

Let $w_1 \supset \Diamond w_2$ be a statement of responsiveness which guarantees that to every situation in which w_1 is true the program responds by making w_2 true. We often wish to complement this statement by requiring that on the other hand, *w_2 will never happen unless preceded by w_1*, *i.e.* the program does not respond unless explicitly requested. This of course is expressible as:

$$\models\ \ w_1\ \mathcal{P}\ w_2,$$

meaning that there is always a w_1 preceding every w_2.

There is however a problem associated with the interpretation of the formal statement above as expressing our intuitive requirement. Assume a situation in which w_1 occurs at t_1 and w_2 indeed follows at t_2, $t_2 > t_1$, and neither w_1 nor w_2 is true between t_1 and t_2. If we try to test the statement: "w_1 precedes w_2" at any t_3, $t_1 < t_3 < t_2$, it will turn out to be *false*, since the first event following t_3 is w_2 rather than w_1. Thus we have to be careful to restrict our statement to only such reference points from which the precedence relation can be safely observed.

Thus a more careful description of the *no-request-no-response* statement is:

$$\models\ \ (at\,\ell_0 \supset w_1\ \mathcal{P}\ w_2)\ \ \wedge\ \ [(w_2 \wedge \bigcirc\!\sim\! w_2) \supset \bigcirc(w_1\ \mathcal{P}\ w_2)].$$

This selects as good reference points from which the precedence of w_1 to w_2 may be observed either the starting point of the computation, or an instant in which w_2 is true and is changing to false in the next instant. In the later case $w_1\ \mathcal{P}\ w_2$ begins to hold only in the next instant.

In most practical cases we have additional information about the behavior of w_1 and w_2 that helps us formulate the requirements in simpler terms. Thus if we knew that once w_1 was raised and not yet answered by a w_2 it stays true until answered, the above problem would not have risen. Instead we could use the simpler

$$\models\ \ (at\,\ell_0 \vee \sim\! w_1)\ \ \supset\ \ w_1\ \mathcal{P}\ w_2.$$

Example:

Let us reconsider the example of the operating system model: an allocator (*granter*) G that allocates a resource between customers (*requesters*) $R_1, \ldots, R_t$. Customer R_i signals its requests by setting r_i to *true*. The allocator G eventually responds by setting g_i to *true*. The customer eventually releases the resource by setting r_i to *false* which the allocator acknowledges by setting g_i to *false*.

This simple communication protocol between a particular customer R_i and the allocator can be specified by the following four invariants:

1. $\models \ (r_i \wedge \sim g_i) \supset \bigcirc\, r_i.$

This says that if r_i is true and g_i is false, meaning that R_i is requesting the resource but has not yet been granted its request, R_i should persist in its request by leaving r_i on for the next instant. Note that we exclude instantaneous response by using the current values of r_i and g_i to determine the *next* value of r_i.

2. $\models \ (r_i \wedge g_i) \supset \bigcirc\, g_i.$

This states that if the resource has been granted to R_i, then the allocator is not allowed to withdraw its grant until the resource is released by R_i, by setting r_i to *false*.

3. $\models \ (\sim r_i \wedge g_i) \supset \bigcirc \sim r_i.$

This states that if the allocator has not yet acknowledged the release of the resource by R_i, then R_i may not issue a new request.

4. $\models \ (\sim r_i \wedge \sim g_i) \supset \bigcirc \sim g_i.$

This states that if the resource is not currently allocated to R_i nor is R_i requesting it, the allocator should not grant the resource to a process which is not requesting it. This is exactly our requirement of *no unsolicited responses* for this case.

These four demands with the additional responsiveness requirement

5. $\models \ r_i \supset \Diamond\, g_i$

6. $\models \ g_i \supset \Diamond \sim r_i$

7. $\models \ \sim r_i \supset \Diamond \sim g_i$

ensure the correct and proper behavior of the system.

The four statements 1–4 above characterize the behavior of the program by immediate transition rules. Since it is not always obvious what are the global consequences of such local constraints, we would prefer to specify them in a more global style. Such specifications can be given by:

(a) $\models \ r_i \supset [r_i\, \mathcal{U}\, (g_i \wedge r_i)]$

(b) $\models \ g_i \supset [g_i\, \mathcal{U}\, (\sim r_i \wedge g_i)]$

(c) $\models \; \sim r_i \supset [\sim r_i \; \mathcal{U} \; (\sim g_i \wedge \sim r_i)]$

(d) $\models \; \sim g_i \supset (r_i \; \mathcal{P} \; g_i)$

which replace 1–7.

Statement (a) says that if r_i is true it will remain true until g_i is granted. Statement (b) says that if the resource is granted it will remain granted until released. Statement (c) says that if the resource has been released it will not be requested again until the release has been acknowledged. Statement (d) says that if g_i is not currently allocated, its next allocation must be preceded by a request. ∎

c. Fair Responsiveness

In many situations we have the precedence of two events ψ_1 and ψ_2, *i.e.*, ψ_1 *precedes* ψ_2 only when two earlier events ϕ_1 and ϕ_2 occurred in the same order, *i.e.* ϕ_1 *precedes* ϕ_2. We will refer to such situations as *conditional precedence.* It is expressible by the statement:

$$\models \; (\phi_1 \; \mathcal{P} \; \phi_2) \supset (\psi_1 \; \mathcal{P} \; \psi_2).$$

This says that if ϕ_1 (strictly) precedes ϕ_2 then ψ_1 will (strictly) precede ψ_2.

Coupled with the implications

$$\models \; \phi_1 \supset \Diamond \psi_1 \quad \text{and} \quad \models \; \phi_2 \supset \Diamond \psi_2 \quad .$$

which ensure responsiveness, the conditional precedence sharpens our committment to *fair responsiveness.* That is, if we interpret $\models \phi_1 \supset \Diamond \psi_1$ and $\models \phi_2 \supset \Diamond \psi_2$ as describing a response ψ_i to a request ϕ_i, then *responsiveness* says that every request will eventually be honored by a response. The *fair responsiveness* establishes a *first-come-first-serve* discipline by ensuring that if ϕ_1 preceded ϕ_2 then the response to ϕ_1, namely ψ_1, will precede the response to ϕ_2, *i.e.* ψ_2.

Example:

Let us consider again the problem of the granter (allocator) and his serviced customers (requesters). We may impose a fairness requirement on his responsiveness obligations by insisting on a first-come-first-serve policy. This would be expressed by:

$$\models \; (r_i \; \mathcal{P} \; r_j) \supset (g_i \; \mathcal{P} \; g_j).$$

This means that if customer R_i placed his request before customer R_j he will be serviced prior to customer R_j. However, we again must be careful to state this only in "quiescent" reference points. For example, if g_j is currently true, while both $r_i = r_j =$ *false*, a situation which may occur just at the end of a granting period to R_j, we certainly cannot promise that g_i will precede g_j.

A reasonable set of reference points is such instants in which g_j is currently *false*. Thus the conditional precedence statement restricted to these observation points is:

$$\models \; (\sim g_j) \supset [(r_i \; \mathcal{P} \; r_j) \supset (g_i \; \mathcal{P} \; g_j)] \quad ∎$$

Example:

Consider a pair of processes where the critical sections $C_1 = \{\ell_2, \ell_3\}$ and $C_2 = \{m_2, m_3\}$ are mutually protected by semaphores:

$$y := 1$$

ℓ_0 : *execute*	m_0 : *execute*
ℓ_1 : *request*(y)	m_1 : *request*(y)
ℓ_2 : *compute*	m_2 : *compute*
ℓ_3 : *release*(y)	m_3 : *release*(y)
ℓ_4 : *go to* ℓ_0	m_4 : *go to* m_0
— P_1 —	— P_2 —

We discussed previously the statement of accessiblity for such a program; namely, that if P_1 is waiting at ℓ_1 it will be eventually admitted into C_1. This ensures only the absence of *infinite overtaking*, *i.e.*, the possibility of P_1 waiting at ℓ_1 forever while P_2 enters its own critical section infinitely often. Yet, can we prevent *overtaking* altogether; *i.e.*, can we prevent P_2 from overtaking P_1 and entering C_2 even though P_1 reached ℓ_1 before P_2 reached m_1?

We may impose fair responsiveness on this situation by requiring that the first process to reach its *request* instruction will be the first to be admitted into its critical section. We may attempt to state this property by:

$$\models \; [(at\,\ell_1 \;\mathcal{P}\; at\,m_1) \supset (at\,C_1 \;\mathcal{P}\; at\,C_2)] \;\;\wedge\;\; [(at\,m_1 \;\mathcal{P}\; at\,\ell_1) \supset (at\,C_2 \;\mathcal{P}\; at\,C_1)].$$

This states that if P_1 gets to ℓ_1 before P_2 gets to m_1 then P_1 will gain access to C_1 before P_2 gets to C_2, and similarly for the dual case in which P_2 gets to m_1 before P_1 gets to ℓ_1.

However we again face the question of appropriate reference points. The statement would certainly not be true if P_2 is currently at C_2. In the above example we may be aided by the location variables in order to select appropriate reference points. One correct specification of fairness of the semaphores in this case is:

$$\models \; [(at\,\ell_1 \wedge at\{m_4, m_0\}) \supset (at\,\ell_2 \;\mathcal{P}\; at\,m_2)] \;\;\wedge\;\; [(at\,m_1 \wedge at\{\ell_4, \ell_0\}) \supset (at\,m_2 \;\mathcal{P}\; at\,\ell_2)].$$

This says that if we are at an instant in which P_1 is already at ℓ_1 while P_2 is both out of C_2 and has not yet arrived at m_1 then P_1 will be admitted to its critical section first, and similarly for the dual case. ∎

One should not be confused by the double appearance of the notion of fairness, once when discussing *fair scheduling* and *fair execution* sequences, and here when discussing *fair responsiveness* as a program property. The concepts are very similar, but previously we assumed fairness as a restriction on execution sequences, since we were interested only in fair execution sequences. Here we consider (and later prove) fairness as a property of the program that gives rise to those sequences. A badly designed program could fail to achieve fairness in responding even when each of the executions we examine is fair as a computation, *i.e.*, the scheduler may be doing its best but the program failed to ensure correct (and timely) response to each request.

Consequently, when we prove that a program has the fair responsiveness property for every proper computation, we *assume* that the computation is scheduled fairly and *prove* that it responds fairly.

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