# ON THE LOOP SWITCHING ADDRESSING PROBLEM

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# On the Loop Switching Addressing Problem

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Abstract.

The following graph addressing problem was studied by Graham and Pollak in devising a routing scheme for Pierce's Loop Switching Network. Let G be a graph with n vertices. It is desired to assign to each vertex $v_i$ an address in $\{0,1,*\}^{\ell}$, such that the Hamming distance between the addresses of any two vertices agrees with their distance in G. Let N(G) be the minimum length $\ell$ for which an assignment is possible. It was shown by Graham and Pollak that $N(G) \leq m_G(n-1)$, where $m_G$ is the diameter of G. In the present paper, we shall prove that $N(G) < 1.09(\lg m_G)n + 8n$ by an explicit construction. This shows in particular that any graph has an addressing scheme of length $O(n \log n)$.

Keywords: addressing scheme, binary tree, graph, Hamming distance, loop switching network.

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## 1. Introduction.

An interesting routing scheme to Pierce's Loop Switching Network [7] was proposed by Graham and Pollak [3,4] (see also [1]). In this scheme, Pierce's network is represented by a graph where vertices stand for the loops, and edges stand for the contacts between loops in the network. The scheme calls for assigning a sequence of ternary symbols to each vertex such that the distances between vertices in the graph are faithfully represented. The combinatorial problem is described below; for a detailed discussion of the connection between Pierce's network and this combinatorial problem, as well as further information on the subject, see references [1,3,4,7].

Throughout our discussion, $G = (V,E)$ will be a connected graph with a set $V$ of vertices, and a set $E$ of undirected edges. A path of length $t$ in $G$ from a vertex $v_i$ to a vertex $v_j$ is a sequence of vertices $v_{k_0}, v_{k_1}, \ldots, v_{k_t}$ such that $v_{k_0} = v_i$, $v_{k_t} = v_j$, and $\{v_{k_{s-1}}, v_{k_s}\} \in E$ for $s = 1,2,\ldots,t$. The distance $d_G(v_i,v_j)$ between vertices $v_i$ and $v_j$ is the minimum length $t$ for which a path of length $t$ from $v_i$ to $v_j$ exists. The diameter of $G$, denoted by $m_G$, is the largest distance between any two vertices in $G$. That is, $m_G = \max\{d_G(v_i,v_j) \mid v_i,v_j \in V\}$.

Let $\Sigma$ be the ternary symbol set $\{0,1,*\}$. (The character "$*$" is a "don't-care" symbol.) The Hamming distance $H$ between elements in $\Sigma$ is defined by $H(1,0) = H(0,1) = 1$, and $H(a,b) = 0$ for all other pairs of $a,b$ in $\Sigma$. For two sequences $\alpha = a_1 a_2 \ldots a_\ell$ and $\beta = b_1 b_2 \ldots b_\ell$ in $\Sigma^\ell$, where $\ell > 0$, their Hamming distance is given by

$$H(\alpha,\beta) = \sum_{1 \le i \le \ell} H(a_i,b_i)_1 .$$

An <u>addressing scheme</u> for a graph $G = (V,E)$ with $n$ vertices is an assignment of a sequence $c(v_i) \in \Sigma^\ell$ to each vertex $v_i$ such that $H(c(v_i),c(v_j)) = d_G(v_i,v_j)$ for all $v_i$, $v_j$ in $V$. The positive integer $\ell$ is called the <u>length</u> of the addressing scheme, and the sequence $c(v_i)$ the <u>address</u> of vertex $v_i$. It is desired to find addressing schemes with small length. Let $N(G)$ be the minimum $\ell$ for which an addressing scheme of length $\ell$ exists for $G$. In [3], it was proved that an addressing scheme always exists (i.e., $N(G) < \infty$), and furthermore, $N(G) \leq m_G(n-1)$. We shall improve this bound by explicitly constructing an addressing scheme. The main results are as follows: (We shall use $\lambda$ to denote the constant $\left(\frac{1}{3}\lg 3 + \frac{2}{3}\lg\frac{3}{2}\right)^{-1} \approx 1.09$.)

<u>Theorem 1</u>. For a graph $G$ with $n$ vertices,

$$N(G) \leq \lambda\, n \lg n + 2n .$$

<u>Theorem 2</u>. For a graph $G$ with $n$ vertices,

$$N(G) \leq \lambda\, n(\lg m_G) + 8n .$$

Note : $\lg$ means logarithm to the base 2.

## 2. Definitions and Preliminaries.

Let $G = (V,E)$ be a (connected, undirected) graph. A path $v_{k_0}, v_{k_1}, \ldots, v_{k_t}$ in $G$ is simple if all the vertices $v_{k_s}$ for $0 \le s \le t$ are distinct, except possibly for $v_{k_0} = v_{k_t}$. A graph $G' = (V',E')$ is called a subgraph of $G$ if $V' \subseteq V$ and $E' \subseteq E$. A subgraph $G' = (V',E')$ is said to be a tree in $G$ if $G'$ is connected and there is no simple path of length > 0 in $G'$ from any vertex $v \in V'$ to itself. A tree $G' = (V', E')$ in $G$ is a spanning tree for $G$ if $V' = V$. For any subset of vertices $V' \subset V$, the diameter of $V'$, written $\mathrm{diam}_G(V')$, is $\max\{d_G(v_i,v_j) \mid v_i,v_j \in V'\}$. In particular, $\mathrm{diam}_G(V) = m_G$. The distance $d_G(v_i, V')$ between a vertex $v_i \in V$ and a subset $V' \subset V$ is defined as $d_G(v_i,V') = \min\{d_G(v_i,v_j) \mid v_j \in V'\}$.

We shall make use of binary trees in our design process. (See for example Knuth [5] for basic definitions regarding binary trees.) Let T be a binary tree with n leaves. Assume the nodes of T are numbered arbitrarily from 1 to 2n-1. The node with number k will be denoted by $r_k$. We will also use the notation $\boxed{i}$ for a leaf numbered i, and (j) for an internal node numbered j. For a node $r_k$, let R(k) be the subset of leaves in T which are descendants of $r_k$. The size of R(k) is called the weight of $r_k$, denoted by w(k). For example, we have $R(1) = \{r_8,r_6,r_9\}$, $R(2) = \{r_2\}$, and $W(1) = 3$, $w(2) = 1$ in Figure 1. The external path length P(T) is defined by the following equation

$$P(T) = \sum_{\text{internal node } r_k} w(k) \quad . \tag{1}$$

The quantity P(T) can alternatively be described as the sum of the distances from the leaves to the root [5]. If $r_i$ and $r_j$ are respectively the leftson and the rightson of $r_k$, we shall write i = leftson(k),

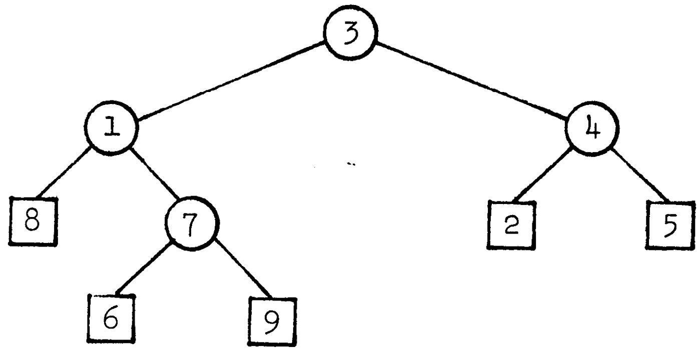


Figure1

$j = \text{rightson}(k)$ ; $k = \text{father}(i) = \text{father}(j)$ ; and $j = \text{brother}(i)$ , $i = \text{brother}(j)$ . As a shorthand, we shall use $\bar{k}$ for father$(k)$ and $k'$ for brother$(k)$ . A binary tree $T$ is said to be <u>weight-balanced</u> if for each internal node $r_k$ ,

$$\frac{1}{3}\, w(k) \le w(\text{leftson}(k)) \le \frac{2}{3}\, w(k) \ ,$$
$$\frac{1}{3}\, w(k) \le w(\text{rightson}(k)) \le \frac{2}{3}\, w(k) \ . \qquad (2)$$

The following result is from [6, Theorem 2].

<u>Lemma 1</u> [Nievergelt and Wong]. If $T$ is a weight-balanced binary tree with $n$ leaves, then the external path length of $T$ satisfies

$$P(T) \le \lambda\, n \lg n \approx 1.09\, n \lg n \ .$$

In a binary tree $T$ , if a leaf $\boxed{i}$ precedes another leaf $\boxed{j}$ in post-order [5], we shall say that $\boxed{i}$ <u>is to the left of</u> $\boxed{j}$ (or $\boxed{j}$ is to the right of $\boxed{i}$ ), and write $\boxed{i} < \boxed{j}$ (or equivalently $\boxed{j} > \boxed{i}$ ). We further extend the relation so that

$$\boxed{i} < r_k \quad \text{if } \boxed{i} < \boxed{j} \text{ for all descendants } \boxed{j} \text{ of } r_k,$$

$$\boxed{i} > r_k \quad \text{if } \boxed{i} > \boxed{j} \text{ for all descendants } \boxed{j} \text{ of } r_k.$$

Clearly, for any leaf $\boxed{i}$ and node $r_k$; either $\boxed{i} < r_k$, $\boxed{i} > r_k$, or $\boxed{i}$ is a descendant of $r_k$; and exactly one of the three relations holds. In Figure 1, we have $\boxed{6} < \boxed{2}$, $\boxed{6} < 4$, and $\boxed{2} > 7$.

## 3. The Construction of a Length $O(n \lg n)$ Addressing Scheme.

### 3.1 The Design Tree.

The key to obtaining an $O(n \lg n)$ scheme is by using a hierarchical design. A design tree $M$ is a pair $(T,f)$ where $T$ is a binary tree with $n$ leaves, and $f$ is a one-to-one mapping from the leaves of $T$ to the vertices of $G$. For notational convenience, we shall number the nodes of $T$ in such a way that the leaves receive numbers 1 to $n$ and leaf $i$ is associated with vertex $v_i$ under $f$. The root of $T$ will be labeled with $2n-1$; and the remaining internal nodes with $n+1$ through $2n-2$ (their actual numbering will be unimportant for $M$).

We now describe an addressing scheme $Z(M)$ corresponding to a given design tree $M$. Every address $c(v_i)$ in $Z(M)$ will consist of $2n-2$ blocks of code, where the k-th block has length $\ell_k$ (to be defined later) and is conceptually associated with the node $r_k$ of $T$. (Note that $r_k$ cannot be the root since $k \neq 2n-1$.) Thus we shall write, for $1 \leq i \leq n$,

$$c(v_i) = c_{i1}\, c_{i2} \cdots c_{i,2n-2} \qquad \text{where } |c_{ik}| = \ell_k \ . \tag{3}$$

By definition, the Hamming distance between two addresses $c(v_i)$ and $c(v_j)$ is equal to the sum of the Hamming distances between corresponding blocks. That is,

$$H(c(v_i),c(v_j)) = \sum_{k=1}^{2n-2} H(c_{ik},c_{jk}) \ . \tag{4}$$

We shall design the code in such a way that in (4), only a few terms will contribute to the sum, other terms being zero. For example, consider the design tree $M$ shown in Figure 2. We shall in fact have

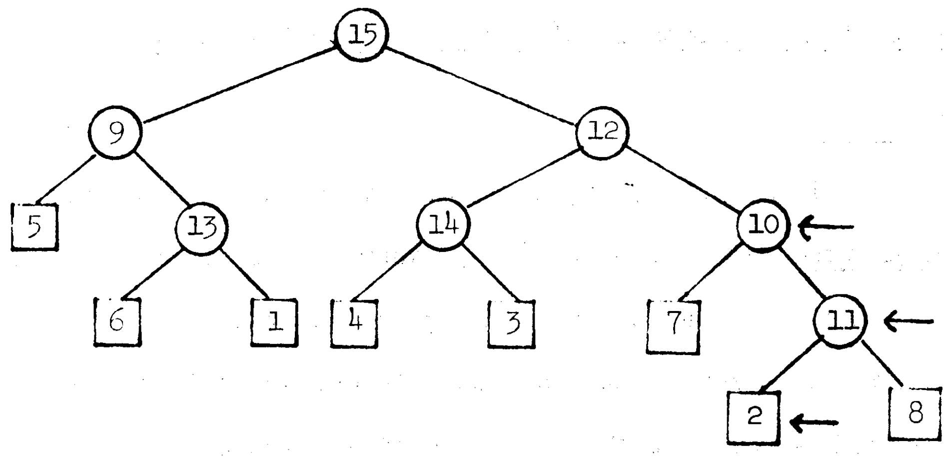


Figure 2. A design tree M with a marked path.

$$H(c(v_3),c(v_2)) = H(c_{3,10},c_{2,10}) + H(c_{3,11},c_{2,11}) + H(c_{3,2},c_{2,2}) \qquad (5)$$

and $H(c_{3,k},c_{2,k}) = 0$ for $k \notin \{10,11,2\}$ . The trick to achieve $H(c(v_3),c(v_2)) = d_G(v_3,v_2)$ is as follows. Define $S(k) = \{f(r_i) \mid r_i \in R(k)\}$ i.e., $S(k)$ is the set of vertices associated with the leaf descendants of $r_k$ . We shall require that,

$$H(c_{3,10},c_{2,10}) = d_G(v_3,S(10)) ,$$

$$H(c_{3,10},c_{2,10}) + H(c_{3,11},c_{2,11}) = d_G(v_3,S(11)) ,$$

$$H(c_{3,10},c_{2,10}) + H(c_{3,11},c_{2,11}) + H(c_{3,2},c_{2,2}) = d_G(v_3,S(2)) . \qquad (6)$$

We can view (6) in the following way. Starting at the <u>lowest common ancestor</u> (lca) of $\boxed{3}$ and $\boxed{2}$ (i.e., the common ancestor of 3 and $\boxed{2}$ farthest from the root), which is' $r_{12}$ , we move down the path $r_{10}$ , $r_{11}$ , to the leaf $r_2$ . Each node $r_k$ encountered along the path, excluding the lca, will add a block of code which creates enough Hamming distance to bring the total up to $d(v_3, S(k))$. An equivalent form of (6) is

$$H(c_{3,k}, c_{2,k}) = d_G(v_3, S(k)) - d_G(v_3, S(\bar{k})) \tag{7}$$

for $k = 10, 11, 2$, and $\bar{k}$ = father(k) .

In general, we want to achieve the following. For $\boxed{i} \lessdot \boxed{j}$ , let node $h_0$ be the lowest common ancestor $\boxed{i}$ i and $c^j_l$, and $h_0, h_1, \ldots, h_t = j$ be the path from node $h_0$ to $c^j_l$, then

$$H(c_{i,k}, c_{j,k}) = d(v_i, S(k)) - d(v_i, S(\bar{k}))$$

for $k = h_1, h_2, \ldots, h_t$ , and $\bar{k}$ = father(k) ;

$$H(c_{i,k}, c_{j,k}) = 0 \quad \text{for all other } k \,. \tag{8}$$

It is easy to verify that (8), if true for all $\boxed{i} \lessdot \boxed{j}$ , will be sufficient to guarantee that $Z(M) = \{c(v_i) \mid 1 \le i \le n)$ , as given by (3), is an addressing scheme, That is, $d_G(v_i, v_j) = H(c(v_i), c(v_j))$ for all $i, j$. We now describe a construction of the $c_{ik}$'s that satisfy (8).

<u>Z(M): The Addressing Scheme Induced by M</u> . For each k, $1 \le k \le 2n-2$ , let

$$\ell_k = \max_{1 \le i \le n} [d_G(v_i, S(k)) - d_G(v_i, S(\bar{k}))] \,. \tag{9}$$

. The block $c_{ik}$ , for $1 \le i \le n$ , has length $\ell_k$ and is given by

$$c_{ik} = \begin{cases} 000 - - - \ldots 0 & \text{if } c^i_l \text{ is a descendant of } r_k \,, \\ *** - - - \ldots * & \text{if } a^i \gtrdot r_k \,, \\ \underbrace{111 - 1}_{\delta}*** \ldots * & \text{with } \delta = d_G(v_i, S(k)) - d_G(v_i, S(\bar{k})) \,, \\ & \text{if } c^i_l \lessdot r_k \,. \end{cases} \tag{10}$$

Finally, form $Z(M) = \{c(v_i) \mid 1 \le i \le n\}$ according to (3). The length of $Z(M)$, denoted by $\tau(M)$, is

$$\tau(M) = \sum_{1 \le k \le 2n-2} \ell_k \quad . \tag{11}$$

To see that $Z(M)$ is actually an addressing scheme, we need only show that (8) is satisfied, For $\boxed{i} < \boxed{j}$, we see from (10) that $H(c_{ik}, c_{jk}) = 0$ unless $\boxed{i} < r_k$ and $\boxed{j}$ is a descendant of $r_k$; in the latter case, $H(c_{ik}, c_{jk}) = d_G(v_i, S(k)) - d_G(v_i, S(\bar{k}))$. But this is exactly as required by (8), q.e.d.

3.2 Criteria for a Good Design Tree.

Let us find out what sort of design tree $M$ will generate a short addressing scheme, Notice that for any $1 \le i < n$, $1 \le k \le 2n-2$, we have

$$d_G(v_i, S(k)) - d_G(v_i, S(\bar{k})) \le \mathrm{diam}_G(S(\bar{k})) \quad . \tag{12}$$

Inequality (12) is valid, since we can concatenate a path from $v_i$ to the nearest point in $S(\bar{k})$, with a path of length at most $\mathrm{diam}_G(S(\bar{k}))$, to reach a vertex in $S(k)$. This tells us that

$$\ell_k \le \mathrm{diam}_G(S(\bar{k})) \quad . \tag{13}$$

An upper bound to $\tau(M)$ is therefore

$$\tau(M) \le \sum_{1 \le k \le 2n-2} \mathrm{diam}_G(S(\bar{k})) = 2 \sum_{n+1 \le k \le 2n-1} \mathrm{diam}_G(S(k)) \, , \tag{14}$$

every internal node being the father of two nodes. This upper bound will in general be $O(n^2)$, as the subset $S(k)$ may have diameter $O(n)$ for many $k$. However, if we insist on two conditions

(i) no two points in $S(k)$ are far apart compared to its size $|S(k)|$, specifically, $\mathrm{diam}_G(S(k)) \le |S(k)|$; and

(ii) the binary tree is weight-balanced,

then (14) would give

$$\tau(M) \le 2 \sum_{n+1 \le k \le 2n-1} |S(k)| = 2\cdot P(T) \le 2\lambda n \lg n \qquad (15)$$

by Lemma 1.

To achieve conditions (i) and (ii), we use the following idea. Let us think of $M = (T,f)$ as a tree built topdown by successively breaking $V$ into smaller parts. From this viewpoint, the tree in Figure 2 is obtained by first dividing (at node 15) $\{v_1,v_2,\ldots,v_8\}$ into $\{v_5,v_6,v_1\}$ and $\{v_4,v_3,v_7,v_2,v_8\}$; each of the two resulting parts are further divided into $\{v_5\}$, $\{v_6,v_1\}$ at node 9, and $\{v_4,v_3\}$, $\{v_7,v_2,v_8\}$ at node 12, respectively. This process is repeated until we have only the singleton sets $\{v_i\}$.

We shall see that in building M in this fashion, it is possible to keep the points in each part close together (condition (i)), and also make the two parts more or less equal in size (condition (ii)) on each decomposition, We shall describe such a method next, and then perform a finer analysis improving the bound given by (15).

## 3.3 Constructing M from a Spanning Tree.

We shall construct a design tree M with the properties (i) and (ii) given in Section 3.2. Choose any spanning tree with edge set A for the graph G, Let us create a new vertex $v_0$ and a new edge $\{v_0,v_1\}$.

We now define a one-to-one mapping $\varphi$ between the edge set of the augmented spanning tree $A' = A \cup \{\{v_0, v_1\}\}$ and the vertex set $V$ (without $v_0$). The mapping $\varphi$ is obtained by regarding $(V \cup \{v_0\}, A')$ as a rooted tree with root $v_0$, and mapping each edge onto its "lower" end point. We shall then number the edges $e_i$ in $A'$ so that $\varphi(e_i) = v_i$. An example of this process is shown in Figure 3.

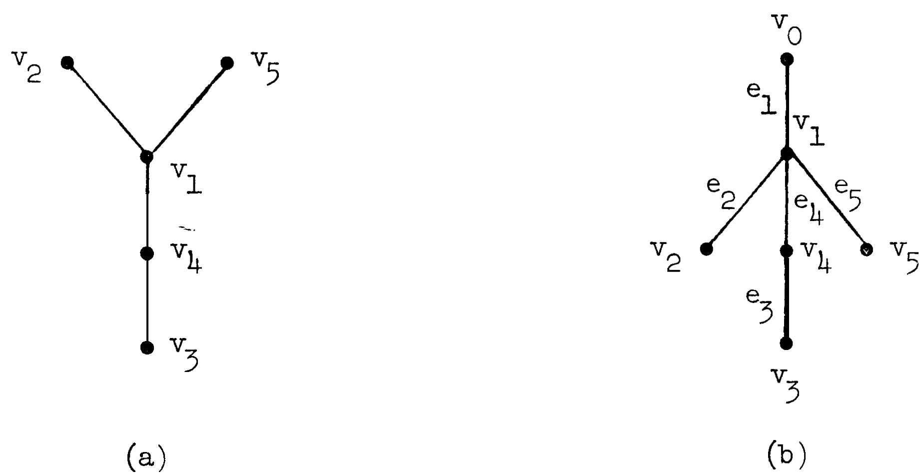


Figure 3. (a) A spanning tree on $V = (v_1, v_2, v_3, v_4, v_5\}$, and (b) the labelling of its edges after augmentation.

Our plan is to construct a binary tree Q by "suitably" splitting the edge set $A'$ into two disjoint subsets, and repeat the process until only one edge remains in each subset. Figure 4(a) shows the binary tree Q that may result from this process when applied to the spanning tree in Figure 3(b). Although the tree Q so constructed is not a design tree on the vertex set, we can easily obtain such a design tree $M_Q$ from

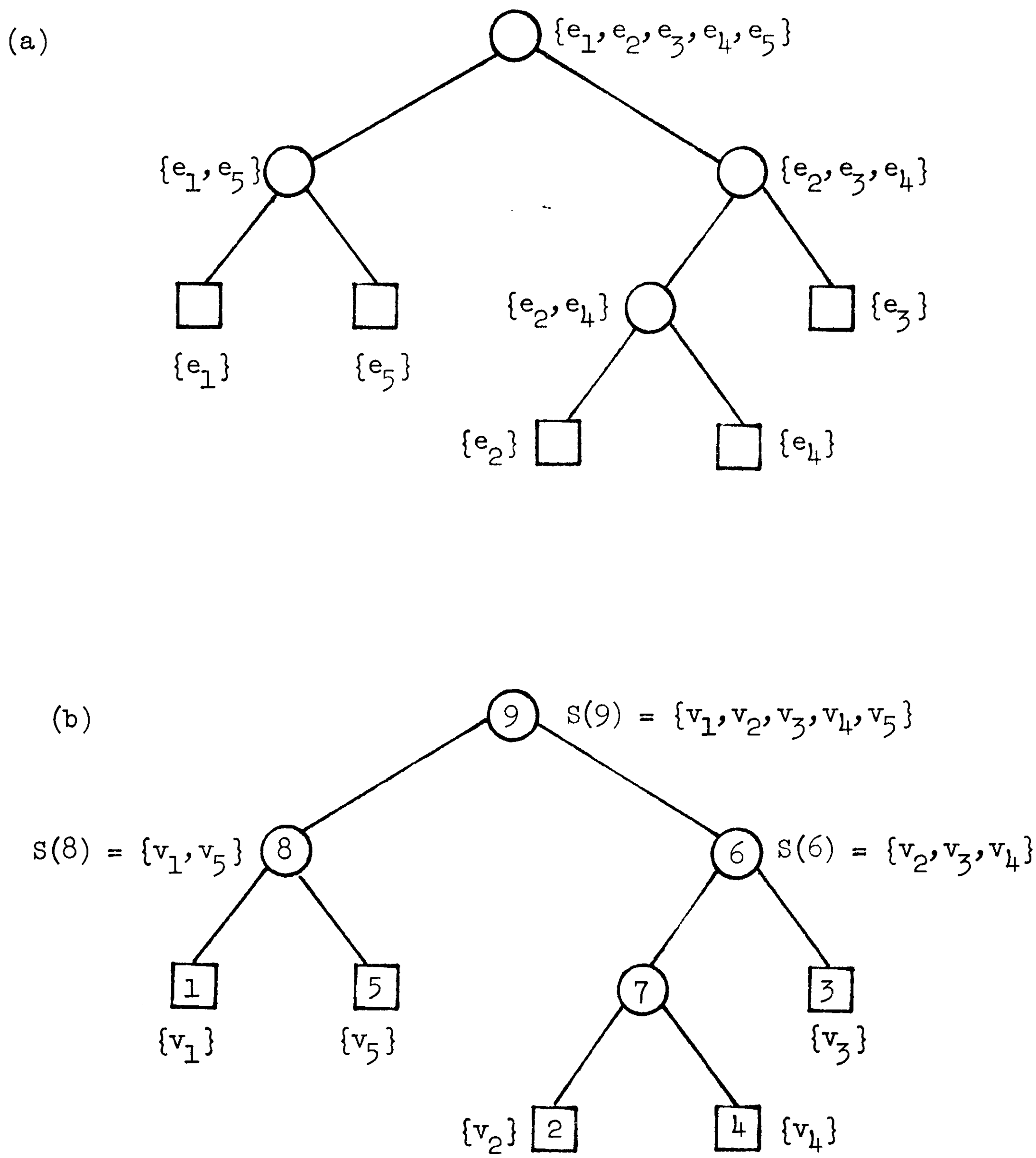


Figure 4. (a) A binary tree $Q$ , and

(b) its associated $M_Q$ .

$Q$ in a natural way via the mapping $\varphi$ , We shall transform $Q$ into $M_Q$ simply by identifying $e_i$ with $v_1$ in the tree $Q$ . The design tree $M_Q$ obtained from the $Q$ in Figure 4(a) is shown in Figure 4(b).

We can now complete our task in two steps, (1) describe the topdown construction of a $Q$ for which $M_Q$ would satisfy conditions (i) and (ii), and (2) analyze the addressing scheme induced by such an $M_Q$ .

(1) Constructing Q . A set of edges $B$ in $G$ is called a tree set if $B$ is the edge set of some tree in $G$ . Two tree sets $B_1$ and $B_2$ is said to--form a decomposition of the tree set $B$ if $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = B$ . Note that, in such a decomposition, there is a unique vertex $v_s$ that is incident to both $B_1$ and $B_2$ , For example, in Figure 3(b), $B = \{e_2, e_4, e_5\}$ is a tree set. We can decompose $B$ into $\{e_2\}$ and $\{e_4, e_5\}$ with $v_1$ being the unique vertex $v_s$ .

A decomposition of $B$ into $B_1$ and $B_2$ is balanced if $\frac{1}{3}|B| \leq B_i \leq \frac{2}{3}|B|$ for $i = 1,2$ . The following lemma is implicit in [2].

Lemma 2 [Chung and Graham]. Any tree set $B$ with $|B| \geq 2$ has a balanced decomposition into two tree sets.

Let us now construct $Q$ by breaking the augmented spanning tree $A'$ into parts successively, using a balanced decomposition at each step. For example, the tree $Q$ shown in Figure 4(a) can be obtained this way from $A'$ in Figure 3(b). Once $Q$ is constructed, we transform it into a design tree $M_Q$ for the vertex set as described previously. It remains to analyze the address length obtained from this tree $M_Q$. To avoid confusion, we use $S(k)$ for the set of vertices associated with node $r_k$ in $M_Q$, and use $B(k)$ to denote the tree set at the corresponding node in $Q$. Clearly, if $S(k) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\}$, then $B(k) = \{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\}$.

(2) Analysis. There are two simple properties of the design tree $M_Q$. Firstly, $M_Q$ is weight-balanced by construction, Secondly, at any node $r_k$ of $M_Q$, $\mathrm{diam}_G(S(k)) \leq |S(k)|$. This is true since any two vertices in $S(k)$ can be connected through at most $|S(k)|$ edges in the tree set $B(k)$. Thus, the two conditions (i) and (ii) in Section 3.2 are satisfied, which implies $\tau(\tilde{M}) \leq 2\lambda n \lg n$. A stronger bound can be obtained, however, by using the following lemma.

Lemma 3. For each node $r_k$ in $M_Q$, and $1 \leq i \leq n$,

$$d_G(v_i, S(k)) - d_G(v_i, S(\bar{k})) \leq 1 + |S(k')| \text{, where } k' = \text{brother}(k) . \tag{16}$$

Proof. Let $v_j$ be a vertex in $S(k)$ closest to $v_i$, i.e.,

$$d_G(v_i, v_j) = d_G(v_i, S(\bar{k})) . \tag{17}$$

If $v_j \in S(k)$, then $d_G(v_i, S(k)) = d_G(v_i, S(\bar{k}))$, and (16) is true. So we can assume that $v_j \in S(k')$.

Let $v_s$ be the unique vertex that is incident to both an edge in $B(k')$ and an edge in $B(k)$. This implies that

$$d_G(v_j, v_s) \leq |B(k')| = |S(k')| . \tag{18}$$

Now, let $\{v_s, v_t\}$ be an edge in $B(k)$ incident with $v_s$ (see Figure 5).

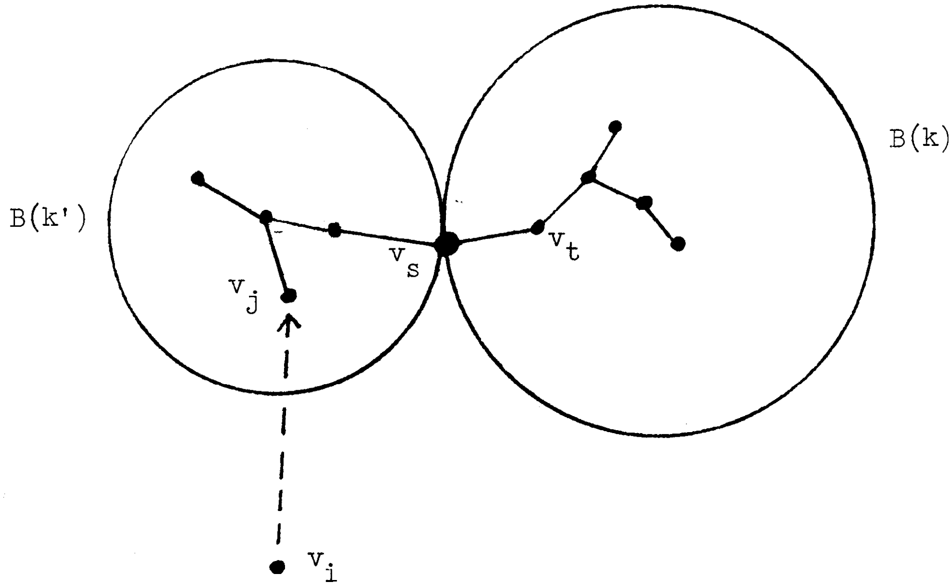


Figure 5

Then,

$$d_G(v_i, v_s) \leq d_G(v_i,v_j) + d_G(v_j,v_s) \leq d_G(v_i,S(\bar{k})) + |S(k')| \quad , \tag{19}$$

$$d_G(v_i, v_t) \leq d_G(v_i,v_s) + 1 < d_G(v_i,S(\bar{k})) + 1 + |S(k')| \quad . \tag{20}$$

Since $\{v_s, v_t\} \in B(k)$ , either $v_s$ or $v_t$ must be in $S(k)$ . Therefore,

$$d_G(v_i, \ldots \leq \max\{d_G(v_i,v_s), d_G(v_i,v_t)\} \quad . \tag{21}$$

Formula (16) follows from (19), (20), and (21). □

Lemma 3 implies that,

$$\ell_k = \max_i \{d_G(v_i,S(k)) - d_G(v_i,S(\bar{k}))\} < 1 + |S(k')| \quad .$$

Therefore

$$\tau(M_Q) = \sum_{1 \le k \le 2n-2} \ell_k \le \sum_{1 \le k \le 2n-2} (1 + |S(k')|)$$

$$= 2n - 2 + \sum_{1 \le k \le 2n-2} |S(k)| \quad . \qquad (22)$$

Making use of the fact that $M_Q$ is weight-balanced and Lemma 1, we obtain after simplification,

$$\tau(M) \le \lambda n \log n + 2n \; .$$

This proves Theorem 1.

3.4 Proof of Theorem 2.

When $m_G$, the diameter of $G$, is substantially smaller than n-1, the addressing scheme we have constructed is better than the bound in Theorem 1 indicates. The key observation is that $\ell_k$ is always no greater than $m_G$, because $\ell_k \le \max_i d_G(v_i, S(k)) \le m_G$. In the analysis of $\tau(M_Q) = \sum \ell_k$, we can thus use $m_G$ to bound $\ell_k$, instead of $1 + |S(k')|$, for some of the nodes $r_k$.

Let X be the set of nodes $r_k$ in $M_Q$ such that $|S(k)| \le m_G$, and $|S(\bar{k})| > m_G$. For each $r_k \in X$, let $J_k = \{r_j \mid r_j$ is a descendant of $r_k$, $r_j \ne r_k\}$. Let $J = \bigcup_{r_k \in X} J_k$. In Figure 6, assume $m_G = 4$, the set X then consists of the nodes marked by arrows, and J is the set of shaded nodes. We shall use inequality $\ell_k < 1 + |S(k')|$ for the nodes $r_k \in J$, and use $\ell_k \le m_G$ for the remaining nodes in deriving a bound for $\tau(M_Q)$.

The following facts will be used in the calculation.

Fact 1. Let q be the number of nodes not in J, then $q < \frac{6n}{m_G}$,

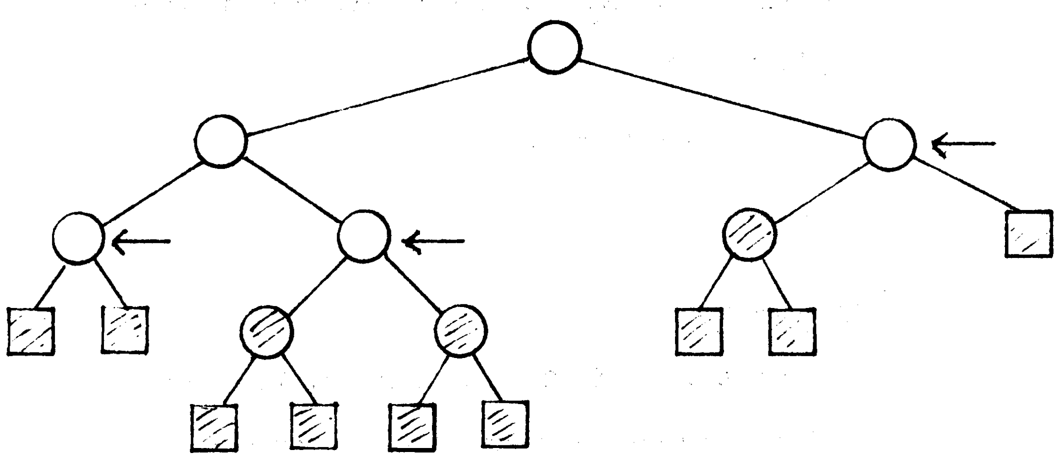

Figure 6. The set of shaded nodes is J .

Proof. $|J| = \sum_{r_k \in X} (2\cdot|S(k)| -2) = 2\left(\sum_{r_k \in X} |S(k)|\right) - 2|X| = 2n - 2|X|$ .

Hence $q = 2n - 1 - |J| = 2|X| - 1$ . Since $|S(k)| \geq \frac{1}{3} |S(\bar{k})| \geq \frac{1}{3} m_G$ for $r_k \in X$ , we have $|X| \leq \frac{n}{\frac{1}{3} m_G}$ . Thus, $q < 2|X| \leq \frac{6n}{m_G}$ . □

Fact 2. Let $r_k \in X$ , then $\sum_{r_j \in J_k} |S(j')| \leq \lambda |S(k)| \lg |S(k)|$ .

Proof. $\sum_{r_j \in J_k} |S(j')| = \sum_{r_j \in J_k} |S(j)|$ . Fact 2 then follows from the fact that the subtree of $M_Q$ rooted at $r_k$ is weight-balanced. □

We can now prove the desired bound as follows:

$$\tau(M_Q) = \sum_{r_k \notin J} \ell_k + \sum_{r_k \in J} \ell_k \leq \sum_{r_k \notin J} m_G + \sum_{r_k \in J} (1 + |S(k')|) .$$

$$= q\, m_G + |J| + \sum_{r_k \in X} \sum_{r_j \in J_k} |S(j')|$$

$$\leq \frac{6n}{m_G} \cdot m_G + 2n + \lambda \sum_{r_k \in X} |S(k)| \lg |S(k)| \tag{24}$$

where we have used Facts 1 and 2 in the last step.

Equation (24) leads to, by using $|S(k)| \leq m_G$ ,

$$\tau(M_Q) \leq 8n + \lambda(\lg m_G) \sum_{r_k \in X} |S(k)|$$

$$= 8n + \lambda(\lg m_G) n .$$

This completes the proof of Theorem 2. □

## 4. Remarks.

In this paper we have given an algorithm which, for a graph with n vertices, constructs an addressing scheme of length $O(n \log n)$. The algorithm can be implemented straightforwardly, and has a $O(n^3)$ running time on a random access machine.

Some slight improvements on our bounds can be obtained by minor modifications of the construction. For example, the $8n$ term in Theorem 2 can be lowered to $4n$, However, we have not found a construction that is guaranteed to give an address of length less than $O(n \log n)$. The very attractive conjecture $N(G) \leq n-1$ of Graham and Pollak [3,4] thus still remains an open problem.

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