ON SPARSE GRAPHS WITH DENSE LONG PATHS

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INTRODUCTION

The following problem was raised by H.-J. Stoss [3] in connection with certain questions related to the complexity of Boolean functions. An acyclic directed graph $G$ is said to have property $\varphi(m,n)$ if for any set $X$ of $m$ vertices of $G$, there is a directed path of length $n$ in $G$ which does not intersect $X$. Let $f(m,n)$ denote the minimum number of edges a graph with property $\varphi(m,n)$ can have. The problem is to estimate $f(m,n)$.

For the remainder of the paper, we shall restrict ourselves to the case $m = n$. We shall prove

\begin{equation}
C_n \log n / \log \log n < f(n,n) < C_2 n \log n
\end{equation}

(where $C_1, C_2, \ldots$ will hereafter denote suitable positive constants). In fact, the graph we construct in order to


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A PRELIMINARY LEMMA

In order to establish the upper bound in (1) we first need the following result.

Lemma. For all \( \delta > 0 \) there exists \( c = c(\delta) \) such that for all \( t \) sufficiently large, there exists a bipartite graph \( B = B(\delta; t) \) with vertex sets \( A \) and \( A' \) so that:

(i) \( |A| = |A'| = t; \)

(ii) \( B \) has at most \( c(\delta)t \) edges;

(iii) If \( X \subseteq A \), \( X' \subseteq A' \) with \( |X| \geq \delta t, |X'| > \delta t \)
    then \( (X, X') = \{ (x, x') : x \in X, x' \in X' \} \) contains an edge of \( B \).

Proof: We use a simple probabilistic argument to show the existence of \( B \). Form a bipartite graph \( \bar{B} \) on the vertex sets \( A \) and \( A' \) with \( |A| = |A'| = t \) by selecting for each \( a \in A \) a random subset \( \bar{B}(a) \subseteq A' \) of cardinality \( d = d(\delta) \) (to be specified later). Call \( \bar{B} \) "bad" if there exists \( X \subseteq A \), \( X' \subseteq A' \), with \( |X| \geq \delta t, |X'| > \delta t \), so that \( (X, X') \) contains no edge of \( \bar{B} \). For fixed \( X \) and \( X' \), the probability that \( \bar{B} \) is bad because of these two subsets is at most

\[
\left( \frac{(1-\delta)t}{d} \right)^{5t} \left( \frac{t}{d} \right)^{5t} < \left( \frac{(1-\delta)t}{t-d} \right)^{d\delta t}.
\]
Hence, the total probability that $B$ is bad is at most

$$\left( \frac{t}{6t} \right)^2 \left( \frac{1-\theta}{t-d} \right)^{d_6 t} < 2^{-t} \left( \frac{1-\theta}{1-d/t} \right)^{d_6 t}$$

A simple computation shows that if $d$ is chosen suitably large, for example, so that

$$(1-\theta^2)^{d_6} < 1/4,$$

then for $t$ sufficiently large this probability is less than 1, and so, a graph $B = B(\theta; t)$ must exist which satisfies the requirements of the lemma.

**CONSTRUCTION OF $G$**

The next step in the proof of (1) is the construction of the directed graph $G$. For large $n$, $G = G(n)$ will have as its vertex set the set $V = \{0,1,...,2^n-1\}$. If $v$ and $m$ are positive integers, then $D_v(m)$ will denote the set $\{v, v+1, ..., v+m-1\} \cap V$. Similarly, $D^*_v(m)$ will denote the set $\{v, v-1, ..., v-m+1\} \cap V$. In general, $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{10}$, will denote suitably chosen fixed positive constants to be specified later. The edge set $E$ of $G$ is formed as follows:

(i) For $v \in V$, the pairs $(v, x)$, $x \in D_{v+1}(4n)$, are in $E$;

(ii) For each $t$ with $n/2 < 2^t < 2^n$ and each $i$ as specified below, a copy of $B(\varepsilon_i 2^t)$ is formed between the vertex sets $A = D_m.2^t(2^t)$ and $A' = D_{m+i}.2^t(2^t)$, $0 \leq m < 2^{n-t}$, where $i = 1, 2, ..., 10$ (or if $i$ cannot assume the value 10 because $(m+10)2^t > 2^n$, then it ranges from 1 to $2^{n-t-m}$). All edges are directed from $x$ to $y$ with $x < y$. 
An elementary calculation shows that

\[ |E| < c_0 n 2^n. \]

**THE UPPER BOUND**

**Theorem 1.** For a suitable \( \varepsilon > 0 \), \( G(n) \) has property \( \mathcal{P}(\varepsilon \cdot 2^n, \varepsilon \cdot 2^n) \) for all sufficiently large \( n \).

**Proof:** The theorem will be proved by a sequence of claims. First we show that \( G(n) \) shares with the graphs \( B(\varepsilon; t) \) the following property.

**Claim 1.** If \( m \geq 2n \) and \( X \subseteq D_x(m), X' \subseteq D_{x+m}(m) \) satisfy \( |X| \geq \varepsilon m, |X'| \geq \varepsilon m \), then \( [X, X'] = \{(x, x') : x \in X, x' \in X'\} \) contains an edge of \( G(n) \).

**Proof of Claim:** Let \( 2^t \leq m/2 < 2^{t+1} \). Thus, \( m/4 < 2^t \) so at most five of the intervals \( D_r \cdot 2^{t}(2^t) \) intersect \( D_x(m) \) and at most five of them intersect \( D_{x+m}(m) \). Since \( |X| > \varepsilon m \), then some \( D_r \cdot 2^{t}(2^t) \) and \( D_{r'} \cdot 2^{t}(2^t) \) have

\[ (3) \ |D_r \cdot 2^{t}(2^t) \cap X| \geq \varepsilon m/5, \quad |D_{r'} \cdot 2^{t}(2^t) \cap X'| > \varepsilon m/5. \]

But we must have \( |r' - r| \leq 10 \) so that by the construction of \( G(n) \) there is a copy of \( B(\varepsilon_1; 2^t) \) between \( D_r \cdot 2^{t}(2^t) \) and \( D_{r'} \cdot 2^{t}(2^t) \). Thus, if \( \varepsilon 1/5 > \varepsilon_1 \) and \( m > 2^t \) then the property of \( B(\varepsilon_1; 2^t) \) guaranteed by the Lemma implies that \( [X, X'] \) contains an edge of \( G(n) \) provided that \( t \) is sufficiently large (which is guaranteed by choosing \( n \) large enough).

This proves the claim. \( \square \)
Next, let us choose an arbitrary fixed set $X$ of vertices with $|X| \leq \varepsilon \cdot 2^n$. The vertices in $X$ will be referred to as the marked vertices of $G$; the remaining vertices of $G$ will be called the unmarked vertices of $G$.

Let us call an unmarked vertex $y \in V$ bad if for some $m \geq 1$ either at least $\varepsilon_3 m$ vertices in $D_y(m)$ are marked or at least $\varepsilon_3 m$ vertices in $D_y^*(m)$ are marked. Otherwise, an unmarked vertex of $G$ is called good.

Claim 2. There are at most $\varepsilon_4 2^n$ bad vertices.

Proof of Claim: Let $y_1$ denote the least unmarked vertex of $G$ (if it exists) for which for some $m_1 \geq 1$, at least $\varepsilon_3 m_1$ vertices in $D_y(1)$ are marked. In general, if $y_1, \ldots, y_k$ and $m_1, \ldots, m_k$ have been defined, let $y_{k+1}$ be the least unmarked vertex of $G$ following $y_k + m_k - 1$ (if it exists) for which for some $m_{k+1} \geq 1$ at least $\varepsilon_3 m_{k+1}$ vertices in $D_{y_k}^*(m_{k+1})$ are marked. We continue this process until it no longer can be applied, so that, say, $y_1, \ldots, y_k$ and $m_1, \ldots, m_k$ have been defined. Similarly, let $y_1^*$ denote the greatest unmarked vertex (if it exists) for which for some $m_1^* \geq 1$, at least $\varepsilon_3 m_1^*$ vertices in $D_{y_1}^*(m_1^*)$ are marked, etc. In this way, we define $y_1^*, \ldots, y_s^*$ and $m_1^*, \ldots, m_s^*$.

It follows from the preceding construction and the definition of a bad vertex that all bad vertices are contained in the set.
Thus, there are at most
\[ M = \sum_{k=1}^{S} m_k + \sum_{k=1}^{S^*} m_k^* \]
bad vertices. However, by our construction there are at least \((\varepsilon_3/2)M\) marked vertices in \(Y\). Since by hypothesis there are at most \(\varepsilon \cdot 2^n\) marked vertices in \(V\) then we have
\[(\varepsilon_3/2)M \leq \varepsilon \cdot 2^n,\]
\[M \leq (2\varepsilon/\varepsilon_3)2^n < \varepsilon 4^2^n,\]
which proves the claim. \(\blacksquare\)

For an unmarked vertex \(x\), let \(P_x(m)\) denote the set of all unmarked vertices in \(D_x(m)\) which can be reached from \(x\) by directed paths which contain only unmarked vertices.

Claim 3. If \(x\) is a good vertex and \(|D_x(m)| = m\) then
\[(4) \quad |P_x(m)| > \varepsilon^5 m\]

Proof of Claim: If \(m \leq 4n\) then since \(x\) is good, at least \((1-\varepsilon_3)m\) vertices in \(D_x(m)\) are unmarked and \(x\) has edges directly to all of them. Suppose \(m > 4n\). Let \(m'\) denote \([m/2]\). Since \(|D_x(m')| = m'\) then by induction \(|P_x(m')| > \varepsilon^5 m'\). Since \(x\) is good then
at most $\varepsilon_3 m$ vertices in $D_x(m)$ are marked. Hence, at most $\varepsilon_3 m$ vertices in $D_{x+m'}(m') \subseteq D_x(m)$ are marked. Since $m' \geq 2n$ and $\varepsilon_5 \geq \varepsilon_2$ then there are edges from $P_x(m')$ to at least $(1-\varepsilon_2)m'$ vertices in $D_{x+m'}(m')$. But at most $\varepsilon_3 m < 3\varepsilon_3 m'$ vertices in $D_{x+m'}(m')$ are marked. Hence, $P_x(m')$ must have edges to at least $(1-\varepsilon_2-3\varepsilon_3)m'$ unmarked vertices in $D_{x+m'}(m')$. Since $1-\varepsilon_2-3\varepsilon_3 > 3\varepsilon_5$ then

$$|P_x(m)| > 3\varepsilon_5 m' > \varepsilon_5 m.$$  

The claim now follows by induction.

In exactly the same way it follows that if $P_x^*(m)$ denotes the set of all unmarked vertices in $D_x(m)$ which are connected to the unmarked vertex $x$ by a directed path containing only unmarked vertices, and $x$ is a good vertex and $|D^*_x(m)| = m$, then-

(4') $$|P_x^*(m)| > \varepsilon_5 m.$$  

Claim 4. Let $x$ and $x'$ be good vertices with $x < x'$. Then $x' \in P_x(2^n)$.

Proof: If $x'-x \leq 4n$ then the claim is immediate since by construction there is an edge from $x$ to $x'$. Assume $x'-x > 4n$. Let $y = [(x+x')/2]$ and let $m = y - x + 1$.  

Consider the intervals $D_{x}(m)$ and $D_{x}^{*}(m)$. Either they are adjacent or they have the single element $y$ in common.

Since $x$ and $x'$ are good then by (4) and (4')

\[(5) \quad |P_{x}(m)| > \varepsilon_{5}m, \quad |P_{x}^{*}(m)| > \varepsilon_{5}m.\]

Since $\varepsilon_{5} > \varepsilon_{2}$ then by Claim 1, there is an edge in $G$ from a vertex of $P_{x}(m)$ to a vertex of $P_{x}^{*}(m)$. Thus, there is a directed path from $x$ to $x'$ containing no marked vertices and the claim is proved. 

The proof of the theorem is now immediate. By Claim 2 there are at least $(1-\varepsilon_{4}-\varepsilon)2^{n}$ good vertices in $G$. By Claim 4 we can form a directed path which contains only unmarked vertices and which contains all the good vertices (since $x'$ can always be chosen to be the next good vertex following $x$). Since $1-\varepsilon_{4}-\varepsilon > \varepsilon$ then the theorem follows (where it is easily seen how the appropriate values of $\varepsilon_{k}$ and $c_{k}$ can be chosen).

THE LOWER BOUND

The following result will establish the lower bound in (1).

**Theorem 2.** Let $H$ be an acyclic directed graph with at most $c_{7}n \log n/\log \log n$ edges where $n$ is a large fixed integer. Then there is a set of at most $n$ vertices of $H$ which hits every directed path of length $n$.

**Proof:** Let us denote the vertex set of $H$ by $V = \{1, 2, \ldots, v\}$. We may assume that $H$ has at least $c_{8}n \log n/\log \log n$ edges. We may also assume that all edges are of the form $(i, j)$ with
i < j. For an edge \( e = (i, j) \) of \( H \), let \( \text{length}(e) \) be defined to be \( j - i \). Partition the edges of \( H \) into classes \( C_0, C_1, \ldots, C_r \) where

\[
C_k = \{ e : 2^{4k \log \log n} < \text{length}(e) < 2^{4(k+1) \log \log n} \}
\]

and \( r = \lfloor \log \frac{v}{4 \log \log n} \rfloor \).

Since \( H \) has at least \( c_8 n \log n / \log \log n \) edges then it follows that \( v > c_9 n^{1/2} \) and \( r > c_{10} \log n / \log \log n \). Hence some class \( C_a \) with \( 0 < a < r \) has at most \( c_{11} n \) elements. Let us delete all vertices in \( H \) incident to any of the edges in \( C_a \). Furthermore, we also delete those vertices \( x \in V \) which satisfy

\[
0 \leq x - m \cdot 2^{4a \log \log n} (1 + 2^{2 \log \log n}) < 2^{4a \log \log n}
\]

for some integer \( m \geq 0 \). This latter step removes at most

\[
\left( 2^{2 \log \log n} - 1 \right)v = o(n)
\]

vertices, since \( v < 2 c_7 n \log n / \log \log n \). Hence we have deleted at most \( c_{12} n \) vertices altogether. However, any directed path remaining has at most

\[
\left( \frac{2^{(4a+2) \log \log n} - 2^{4a \log \log n}}{2^{4(a+1) \log \log n}} \right)v = o(n)
\]
edges, since we cannot go more than \(2^{(4a+2) \log \log n - 2^{4a} \log \log n}\) steps without using an edge whose length exceeds \(2^{4a} \log \log n\); and the length of such an edge actually exceeds \(2^{4(a+1) \log \log n}\). This proves the theorem. \(\Box\)

By using a different partition of the edges of \(H\), namely, into the classes \(C'_0, ..., C'_r\), where

\[C'_k = \{ e : 2^{c_{13}^k} \leq \text{length}(e) < 2^{c_{13}^k(k+1)} \}\]

for a suitable constant \(c_{13}\), we can establish the following result.

**Theorem 3.** If \(c_{14}\) is sufficiently large then any graph \(G\) on \(c_{14}n\) vertices having property \(\mathcal{P}(n,n)\) must have at least \(c_{15}n \log n\) edges.

The graphs \(G(n)\) used in Theorem 1 show that the result in Theorem 3 is best possible to within constant factors.

**SOME RELATED QUESTIONS**

We now consider several problems for ordinary (undirected) graphs. Let \(F_e(n,n)\) (resp., \(F_v(n,n)\)) denote the smallest integer for which there is a graph with \(F_e(n,n)\) (resp., \(F_v(n,n)\)) edges so that with the deletion of any \(n\) of its vertices there still remains a connected component of \(n\) edges (resp., vertices). We shall prove by probabilistic methods that

\[(6) \quad F_e(n,n) < c_{16}n, F_v(n,n) < c_{17}n.\]

The method we use is the same as that in the work of Erdős and Rényi [1], [2]. It turns out that almost all graphs have the desired property.
Theorem 4. For every $\varepsilon > 0$ there is a $c = c(\varepsilon)$ so that all but $\Theta\left(\binom{(2+\varepsilon)n}{2+\varepsilon}\right)$ graphs $G$ with $(2+\varepsilon)n$ vertices and $cn$ edges have the property that after the omission of any $n$ of its vertices, a connected component of at least $n$ vertices remains.

Proof: It suffices to show that if $n$ vertices are omitted and the remaining $n(1+\varepsilon)$ vertices are split into two classes $S_1$ and $S_2$ with $|S_1| > \varepsilon n$, $|S_2| > \varepsilon n$, then there is at least one edge joining a vertex of $S_1$ to a vertex of $S_2$.

Consider a random graph $G$ on $(2+\varepsilon)n$ vertices and $cn$ edges (where $c$ will be specified later). There are $\binom{(2+\varepsilon)n}{n}$ ways that $n$ vertices of $G$ can be deleted. The remaining $n(1+\varepsilon)$ points can then be split into two sets $S_1$ and $S_2$ in at most $2^{n(1+\varepsilon)}$ ways. Thus, the total number of splittings is at most

$$ \left(\binom{(2+\varepsilon)n}{n}\right)^2 n^{(1+\varepsilon)} < 2^{(2+\varepsilon)n} 2^{n(1+\varepsilon)} < 2^{3(1+\varepsilon)n}. $$

Between $S_1$ and $S_2$ there are at least $\varepsilon n^2$ potential edges. The probability that none of these edges actually occurs in $G$ is less than $(1 - \frac{c}{(2+\varepsilon)n})^{\varepsilon n^2}$. Thus, if $c$ is chosen so that

$$ 2^{3(1+\varepsilon)n} \left(1 - \frac{c}{(2+\varepsilon)n}\right)^{\varepsilon n^2} \to 0 $$
as $n \to \infty$ then we easily see that almost all graphs cannot be split in such a way.

Since

$$
\left(1 - \frac{c}{(2+\varepsilon)n}\right)^{\varepsilon n^2} \to e^{-(\frac{\varepsilon c}{2+\varepsilon})n}
$$

then for $c$ large enough, e.g., $c > 18(\varepsilon + \varepsilon^{-1})$,

$$
e^{-(\frac{\varepsilon c}{2+\varepsilon})n} < e^{-3(1+\varepsilon)n}
$$

and (7) holds. This proves the theorem. ■

The other half of (6) is proved in a similar way. It would be interesting to determine the best possible value of $c$ but this does not seem to be too easy.

We mention here the undirected analogue of (1). Let $g(n,n)$ denote the smallest integer for which there is an undirected graph of $g(n,n)$ edges so that if we omit any $n$ of its vertices then there always remains a path of length $n$. We believe

$$
g(n,n) \frac{n}{n} \to \infty, \quad g(n,n) \frac{n}{n \log n} \to 0
$$

as $n \to \infty$ and hope to return to this question in finite time.

A related question is the following: Consider random graphs on $n$ vertices and $Cn$ edges. Is it true that for large $C$ almost all of these graphs have a path of length $n(1-\varepsilon)$? It is known [4] that almost all graphs on $n$ vertices and $(\frac{3}{2} + \varepsilon) n \log n$ edges are Hamiltonian.
It is possible to introduce another parameter into these questions. Let $F_v(t;n,n)$ denote the smallest integer for which there is a graph with $t$ vertices and $F_v(t;n,n)$ edges having the property that if any $n$ vertices are deleted there still remains a connected component with at least $n$ vertices. If $t/n \to c > 2$ then $F_v(t;n,n)/n \to A(c)$ where $A(c) \to \infty$ as $c \to 2$. (The behavior of $F_e(t;n,n)/n$ is similar). We could also omit edges instead of vertices but leave the formulation of these questions to the reader.

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3. D. E. Knuth (personal communication)