A FINITE BASIS THEOREM REVISITED

BY

DAVID A. KLARNER

STAN-CS-73-338

FEBRUARY 1973

COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY

A FINITE BASIS THEOREM REVISITED

David A. Klarner
Computer Science Department
Stanford University

Abstract

Let S denote a set of k-dimensional boxes each having integral sides. Let $\Gamma(S)$ denote the set of all boxes which can be filled completely with translates of elements of S . It is shown here that S contains a finite subset B such that l?(B) = $\Gamma(S)$. This result was proved for $k = 1,2$ in an earlier paper, but the proof for $k > 2$ contained an error.

This research was supported in part by the National Science Foundation under grant number GJ-992, and the Office of Naval Research under contract number N-00014-67-A-0112-0057 NR 044-402.

Let N and P denote the sets of non-negative and positive integers respectively, and let N^k and P^k denote the sets of k-tuples of elements of these sets for each $k \in P$. Natural ordering and ordering by division in P may be extended to P^k in the usual way: thus, $(a_1,\ldots,a_k) \le (b_1,\ldots,b_k)$ just when $a_i \le b_i$ for $i = 1,\ldots,k$, and $(a_1,\ldots,a_k) | (b_1,\ldots,b_k)$ just when $a_i | b_i$ for $i = 1,\ldots,k$. We shall use Dedekind's notation $a \wedge b$ for the greatest common divisor of $a,b \in P$, and write $\bigwedge A$ for the greatest common divisor of a non-empty subset $A \subset P$. Also, $a \vee b$ denotes the least common multiple of $a,b \in P$, while $\bigvee A$ denotes the least common multiple of a non-empty, finite subset $A \subseteq P$. These concepts and notations extend in an obvious way to P^k ordered by division.

Let $[\bar{a},\bar{b}]$ denote the interval in P^k, ordered naturally, having lower end $\bar{a}$ and upper end $\bar{b}$; that is, $[\bar{a},\bar{b}] = \{\bar{x}: \bar{x} \in P^k \text{ and } \bar{a} \le \bar{x} \le \bar{b}\}$, and this set is non-empty only when $\bar{a} \le \bar{b}$. Also, let $\bar{1} = (1,\ldots,1)$ denote the k-tuple of 1's in P^k. The interval $[\bar{1},\bar{d}]$ is called a k-<u>dimensional box</u> with dimensions $\bar{d} \in P^k$ which we denote $\bar{d}$, and the interval $[\bar{1}+\bar{t},\bar{d}+\bar{t}]$ with $\bar{t} \in N^k$ is called a <u>translate</u> of the box $\bar{d}$. A set $\mathcal{A}$ of sets is said to <u>pack</u> a set A just when some subset of $\mathcal{A}$ is a partition of A. The <u>closure</u> of a set S of k-dimensional boxes is defined to be the set $\Gamma(S)$ of all k-dimensional boxes which can be packed with the set of all translates of all elements of S. It is easy to see that Γ is a closure operation; that is, $S \subseteq \Gamma(S) = \Gamma(\Gamma(S))$ for all sets S, and $\Gamma(R) \subseteq \Gamma(S)$ for all $R \subseteq S$. The finite basis theorem for box packing which was discussed in [1] is as follows; Every set S of k-dimensional boxes contains a finite subset B such that $\Gamma(B) = \Gamma(S)$.

Unfortunately, the proof given in [1] breaks down for $k > 2$, and it is the purpose of this paper to give a correct extension of the proof given for $k = 1$ and 2. In an effort to discover a relationship between this theorem and known results concerning basis theorems in lattice theory, we have formulated some of our lemmas in a general setting. It appears that the situation involving box-packing is outside what is already known generally about closure operators.

A sequence $(x_n : n \in P)$ of elements of a lattice L is said to be <u>stable</u> just when $x_n \wedge x_{n+1} = x_n \wedge x_{n+2} = \ldots$ for all $n \in P$. We record the obvious fact that stability of a sequence is a property inherited by subsequences.

<u>Lemma 1</u>. Subsequences of stable sequences are stable.

A lattice L is said to be <u>locally finite</u> just when the interval $\{y \in L : x \leq y \leq z\}$ is finite for all $x, z \in L$. Important examples of a locally finite lattices are the set P^k of k-tuples of positive integers ordered by division and P^k ordered naturally. Later we shall require the fact that every infinite sequence of elements of P^k ordered by division contains an infinite stable subsequence. This fact is implied by the following result.

<u>Lemma 2</u>. Every infinite sequence of elements of a locally finite lattice with a least element contains an infinite stable subsequence.

<u>Proof</u>. We use the König infinity lemma which asserts that an infinite rooted tree all of whose vertices have finite degree has an infinite path starting at the root of the tree. In our application, the vertices of the

tree will be certain (possibly finite) subsequences of a given sequence $\bar{x} = (x_n : n \in P)$ whose elements belong to a locally finite lattice with a least element ℓ. First, $\bar{x}$ is designated the root, and then the rest of the tree is defined by specifying the vertices joined below any given vertex $\bar{y} = (y_n : n \in P)$ in the tree. For each d in the (necessarily finite) interval $[\ell, y_1] = \{z : \ell < z \leq y_1\}$, let $s(\bar{y}, d)$ denote the subsequence of $\bar{y}$ consisting of all elements y_i with $i > 2$ such that $y_1 \wedge y_i = d$. The vertices joined below $\bar{y}$ in the tree are the non-empty sequences $s(\bar{y}, d)$ for all $d \in [\ell, y_1]$. Thus, every vertex in the tree has finite degree. Also, since every term of $\bar{x}$ is the initial term of some sequence which is a vertex in the tree, the tree is infinite. Applying the König infinity lemma, we conclude that there exists an infinite path $(\bar{x}_n : n \in P)$ in the tree. Let s_n denote the first term of $\bar{x}_n$ for all $n \in P$, then $\bar{s} = (s_n : n \in P)$ is a stable subsequence of $\bar{x}$. To see this, recall that $\bar{x}_{n+k}$ is a subsequence of $\bar{x}_n$ with s_n deleted for all $k \in P$, and $s_n \wedge y$ is the same for all terms y of $\bar{x}_n$. Hence, $s_n \wedge s_{n+1} = s_n \wedge s_{n+2} = \cdots$ for all $n \in P$. This completes the proof.

Now we establish certain properties possessed by the closure operator Γ. In fact, what we want to prove can be proved in a wider context, and since it doesn't cost us any extra space, we do this. To see that Γ (as defined for box packing) has the property assumed in our next lemma, note that if translates of all of the boxes in a set X are used to pack a box y, then none of the elements of X is larger than y.

Lemma 3. Let S denote a set of elements belonging to a locally finite lattice L . Let Γ denote a closure operator on L having the property that if $y \in L$, $X \subseteq L$, and $y \in \Gamma(X)$, then $y \in \Gamma\{x \in X: x \leq y\}$. Let

(1) $B(S) = \{s \in S$: if $s \in \Gamma(X)$ for some $X \subseteq S$, then $s \in X]$.

Then $\Gamma(B(S)) = I'(S)$, and $B(T) = T$ for all $T \subseteq B(S)$.

Proof. Let $B = B(S)$. If $S \backslash \Gamma(B) = \emptyset$, then $S \subseteq \Gamma(B)$ which implies $r(s) \subseteq \Gamma(\Gamma(B)) = r(B) \subseteq r(S)$ because $B \subseteq S$. That is, $r(s) = r(B)$. Now suppose $S \backslash \Gamma(B) \neq \emptyset$, and select $y \in S \backslash \Gamma(B)$ so that all $x \in S$ with $s < y$ are elements of $r(B)$. Such a minimal element y exists in $S \backslash \Gamma(B)$ because L is locally finite. Since $y \notin \Gamma(B)$, we have $y \notin B$, so there exists a subset $X \subseteq S$ with y&(X) , but $y \notin X$. Let $z = \{x \in X: x \leq y\}$, then we have $y \in \Gamma(Z)$ and $y \notin Z$. Also, $Z \subseteq I'(B)$ because y is minimal in $S \backslash \Gamma(B)$. This means $y \in \Gamma(Z) \subseteq \Gamma(\Gamma(B)) =$ l?(B) $\subseteq I'(S)$ because $B \subseteq S$; that is, $y \in \Gamma(B)$, a contradiction.

Finally, suppose $T \subseteq B$, then elements $t \in T$ have the property possessed by all elements of B ; namely, $t \in \Gamma(X)$ for some $X \subseteq S$ implies $t \in X$, and this is true in particular for all $X \subseteq T$. We conclude that $T = \{t \in T$: if $t \in \Gamma(X)$ for some $X \subseteq T$, then $t \in X\} = B(T)$. This completes the proof.

Lemma 4. Let $\bar{b}_n \in P^k$ with $\bar{b}_n = (b_{n_1}, \ldots, b_{n_k})$ for $n = 1, \ldots, 2^k$, and suppose $(\bar{b}_n: n = 1, \ldots, 2^k)$ is stable. Let $\beta_j = b_{2^k-1,j} \wedge b_{2^k,j}$ for $j = 1, \ldots, k$. Then there exists an integer p such that $\Gamma\{\bar{b}_1, \ldots, \bar{b}_{2^k}\}$ contains all boxes having dimensions $(q_1\beta_1, \ldots, q_k\beta_k)$ with $q_1, \ldots, q_k \geq p$.

Proof. Let $\mu_i(r) = b_{1i} \vee \cdots \vee b_{2^r,i}$ for $i,r = 1,\ldots,k$. We shall show by induction on j that there exists a number p_j such that every box having dimensions

(2) $$(q_1\beta_1,\ldots,q_j\beta_j, \mu_{j+1}(j),\ldots,\mu_k(j))$$

with $q_1,\ldots,q_j \geq p_j$ is an element of $\Gamma\{\bar{b}_1,\ldots,\bar{b}_{2^j}\}$.

For $j = 1$, boxes having dimensions

$$(b_{11}x + b_{21}y, b_{12}\vee b_{22},\ldots,b_{1k}\vee b_{2k})$$

for all $x,y\in N$ are elements of $\Gamma\{\bar{b}_1,\bar{b}_2\}$. But, there exists an integer p_1 such that $q_1\beta_1 \in \{b_{11}x + b_{21}y : x,y\in N\}$ for all $q_1 \geq p_1$ because $b_{11}\wedge b_{21}$ divides β_1. Thus, the claim is true for $j = 1$.

Now we suppose the statement is true for some $j > 1$, and then prove it for $j+1$. Let $\mu'_i(r) = b_{2^r+1,i} \vee \cdots \vee b_{2^{r+1},i}$, and note that the statement involving (2) also applies to the stable sequence $(b_n : n = 2^j+1,\ldots,2^{j+1})$. Thus, there exists a number p'_j such that every box having dimensions

(3) $$(q_1\beta_1,\ldots,q_j\beta_j, \mu'_{j+1}(j),\ldots,\mu'_k(j))$$

with $q_1,\ldots,q_j \geq p'_j$ is an element of $\Gamma\{\bar{b}_{2^j+1},\ldots,\bar{b}_{2^{j+1}}\}$. Boxes having dimensions given by (2) and (3) have boxes in their closure with dimensions

(4) $$(q_1\beta_1,\ldots,q_j\beta_j, x\mu_{j+1}(j) + y\mu'_{j+1}(j), \mu_{j+2}(j)\vee\mu'_{j+2}(j), \ldots, \mu_k(j)\vee\mu'_k(j))$$

for all $q_1,\ldots,q_j \geq \max\{p_j,p'_j\}$ and all $x,y\in N$. Now we observe that $\mu_{j+1}(j)\wedge\mu'_{j+1}(j)$ divides β_{j+1}, and there exists an integer $p_{j+1} \geq \max\{p_j,p'_j\}$ such that $q_{j+1}\beta_{j+1} \in \{x\mu_{j+1}(j) + y\mu'_{j+1}(j) : x,y\in N\}$ for all $q_{j+1} \geq p_{j+1}$. Also, note that $\mu_t(j) \vee \mu_t'(j) = \mu_t(j+1)$ by definition for $j = 1,\ldots,k$. Thus, we have shown that (2) holds for $j+1$ if it holds for j . This completes the proof.

Lemma 5. Let $\bar{s} = (\bar{s}_n : n\in P)$ denote a stable sequence of k-dimensional boxes, and let $S = \{\bar{s}_n : n\in P\}$. Then B(S) is finite.

Proof. The proof is by induction on the dimension k of the boxes. First, we prove the statement for $k = 1$. Let $\bar{s} = (s_n : n\in P)$ denote a stable sequence of 1-dimensional boxes (that is, $s_n \in P$ for all $n\in P$), and let $S = \{s_n : n\in P\}$, and suppose B(S) is infinite. Let $\bar{b} = (b_n : n\in P)$ denote the elements of B(S) ordered according to their sequential ordering in $\bar{s}$. Since $\bar{s}$ is stable, this is also true of $\bar{b}$. Furthermore, since B(S) is infinite, $\bar{b}$ tends to infinity. The closure of B(S) contains the closure of $\{b_1,b_2\}$ which is $\{b_1x + b_2y : x,y\in N\}$, but this set contains all large multiples of $b_1 \wedge b_2$. Since every element of' B(S) is a multiple of $b_1 \wedge b_2$, and since $\bar{b}$ tends to infinity, it follows that there exists $j\in P$ such that $b_j \in \Gamma\{b_1,b_2\}$, but $b_j \notin \{b_1,b_2\}$. This contradicts the definition of B(S) , so B(S) must be finite when $k = 1$.

Now suppose there exists some $k\in P$ such that the statement is false; furthermore, suppose k is minimal, and $k > 1$. Let $\bar{s} = (\bar{s}_n : n\in P)$ denote a sequence of k-dimensional boxes, let $S = \{\bar{s}_n : n\in P\}$, and suppose B(S) is infinite. Also, let $\bar{b} = (\bar{b}_n : n\in P)$ denote the elements of B(S) ordered by their sequential ordering in $\bar{s}$, and let $b_n = (b_{n1},\ldots,b_{nk})$ for all $n\in P$. Note that each of the sequences $(b_{ni} : n\in P)$ for $i = 1,\ldots,k$ tends to infinity. If this were not true, say for $i = k$, we could find an infinite subsequence $\bar{c} = (c_n : n \epsilon P)$ of $\bar{b}$ with $\bar{c}_n = (c_{n1}, \ldots, c_{nk})$ such that $c_{1k} = c_{2k} = \cdots$. Since $C = \{\bar{c}_n : n \epsilon P\} \subseteq B(S)$, we have C -- $B(C)$ infinite. Also, $\bar{c}$ is a stable sequence since $\bar{c}$ is a subsequence of a stable sequence. Now let $\bar{c}_n^{\,*} = (c_{n1}, \ldots, c_{n,k-1})$ for all $n \epsilon P$, let $\bar{c}^{\,*} = (\bar{c}_n^{\,*} : n \epsilon P)$, and let $C^* = \{\bar{c}_n : n \epsilon P\}$. Evidently, $\bar{c}^{\,*}$ is an infinite stable sequence, and $B(C^*) = C^*$ is infinite. Since $\bar{c}^{\,*}$ has dimension $k-1$, this contradicts the minimal property of k . Thus, each of the sequences $(b_{ni} : n \epsilon P)$ for $i = 1, \ldots, k$ tends to infinity.

According to Lemma 4, there exists an integer p such that $\Gamma\{\bar{b}_1, \ldots, \bar{b}_{2^k}\}$ contains every box having dimensions $(q_1\beta_1, \ldots$ [illegible] with $q_1, \ldots, q_k \geq p$. Thus, there exists $\bar{b}_j \epsilon B(S)$ such that $\bar{b}_j \not\epsilon \{\bar{b}_1, \ldots, \bar{b}_{2^k}\}$ but $\bar{b}_j \epsilon \Gamma\{\bar{b}_1, \ldots, \bar{b}_{2^k}\}$. This contradicts the definition of $B(S)$, so the proof is complete. Now we are ready to prove our main result.

Theorem. Let S denote a set of k-dimensional boxes, then there exists a finite subset B of S such that $I'(B) = \Gamma(S)$. In fact, one can take $B = B(S)$.

Proof. We showed that $l?(S) = \Gamma(B(S))$ in Lemma 3, so it is enough to prove that B(S) is finite. Suppose B(S) is infinite. Then we can form an infinite stable sequence $\bar{t} = (\bar{t}_n : n \epsilon P)$ using distinct elements of B(S) . B-u-t $T = \{\bar{t}_n : n \epsilon P\} \subseteq B(S)$, so $T = B(T)$ by Lemma 3. But $B(T)$ is finite according to Lemma 5, so we have a contradiction and the theorem is proved.

The construction given in Lemma 4 involves packing a large box by cutting it with a plane into two smaller boxes, then the smaller boxes are treated in a similar way. We call this simple packing. It is interesting to note that a slight alteration of the foregoing argument yields the result that P(S) contains a finite subset T such that every element of $\Gamma(S)$ can be simply packed with translates of elements of T . We leave the proof as an exercise.

Reference

[1] D. A. Klarner and F. Göbel, "Pacing Boxes with Congruent Figures," Indag Math., 31 (1969), pp. 465-472.