ON A MODEL FOR COMPUTING ROUND-OFF ERROR OF A SUM

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Given real numbers \( a_1, a_2, \ldots, a_n \) we are interested in the classic problem of the error in computing \( S = \sum_{i=1}^{n} a_i \) when the sum is computed by \( \tilde{S}_0 = \sum_{i=1}^{n} a_i^* \) where \( a_i^* \) is the nearest integer to \( a_i \). We shall first study this error as a function of a \( A \) shift, i.e., when all numbers \( a_i \) are each shifted \( A \) and then rounded:

\[
(1) \quad S - nA = \sum_{i=1}^{n} (a_i - A)
\]

\[
(2) \quad \tilde{S}_A - nA = \sum_{i=1}^{n} (a_i^* - A)
\]

We will then let \( A \) become a random variable that can take on uniformly any value in the interval \( -\frac{1}{2} \leq A \leq +\frac{1}{2} \). Different choices of \( A \) give rise to different rounding errors \( \tilde{S}_A - s \) and the variance of the distribution of \( \tilde{S}_A - s \) can be used to measure the variability of the rounding error due to the random selection of the origin of the real numbers \( a_i \) with respect to that of the computer.

The cumulative error from (1) and (2) is

\[
(3) \quad \tilde{S}_A - s = \sum_{i=1}^{n} [(a_i^* - A)^* - (a_i - A)]
\]

Let \( f_i \) be the positive fractional part of \( a_i \) and let \( a_i \) be the largest integer not exceeding \( a_i \), i.e.,

\[
(4) \quad a_i = \alpha_i + f_i
\]
Denoting by \( r_i \) the error of the \( i^{th} \) term, we have

\[
(5) \quad r_i = [(a_i - A) - (a_i - A)] = \begin{cases} 
1 - (f_i - \Delta) & \text{if } -\frac{1}{2} < A < -\frac{1}{2} + f_i \\
-(f_i - \Delta) & \text{if } -\frac{1}{2} + f_i < A < \frac{1}{2} 
\end{cases}
\]

To prove the above, we note that \( f_i - \Delta = (a_i - \Delta) + a_i \). If

\(-\frac{1}{2} < f_i - \Delta < \frac{1}{2}\) \quad \text{then} \quad (a_i - \Delta) \quad \text{is rounded to } a_i. \quad \text{Hence} \quad a_i - \Delta \quad \text{is rounded down if } -\frac{1}{2} + f_i < A \quad \text{otherwise rounded up.}

Denoting expected value by \( E \), we have by direct evaluation

\[
(6) \quad E(r_i) = \int_{-\frac{1}{2}}^{\frac{1}{2}} r_i d\Delta = 0
\]

Assume \( f_i < f_j \)’, then

\[
E(r_i r_j) = \int_{-\frac{1}{2}}^{\frac{1}{2}} r_i r_j d\Delta + \int_{-\frac{1}{2}}^{\frac{1}{2}} r_i r_j d\Delta + \int_{-\frac{1}{2}}^{\frac{1}{2}} r_i r_j d\Delta
\]

\[
= \int_{-\frac{1}{2}}^{\frac{1}{2}} (f_i f_j - \Delta(f_i + f_j) + \Delta^2) d\Delta
\]

\[
+ \int_{-\frac{1}{2}}^{\frac{1}{2}} [(-f_i - f_j) + 2\Delta] d\Delta
\]

\[
+ \int_{-\frac{1}{2}}^{\frac{1}{2}} [-f_i + \Delta] d\Delta
\]

Performing indicated integration yields:

\[
(7) \quad E(r_i r_j) = \frac{1}{2} \left[ |f_j - f_i| - |f_j - f_i| + \frac{1}{6} \right]
\]
which is one-half the \(2^{nd}\) order Bernoulli Polynomial in \(|f_j - f_i|\). For \(f_j < f_i\) we also get (7). Note that the individual errors \(r_i\) and \(r_j\) are not independent of one another.

It now follows that

\[
(9) \quad E(S) = S
\]

\[
(10) \quad E(S-S)^2 = E\left(\sum_{i=1}^{n} \sum_{j=1}^{n} r_ir_j\right) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[|f_i - f_j|^2 - |f_i - f_j| + \frac{1}{6}\right]
\]

The usual value of variance, \(E(S-S)^2 = n/12\), will result if we further assume \(f_i\) are independently drawn from uniform distributions on \([0 < f_i < 1]\).

**Theorem:** If the fractional parts of all \(a_i\) are equal to each other, then each term of (10) is maximum for \(0 < f_i < 1\) and

\[
(11) \quad \text{Max} \ E(S-S)^2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1}{6}\right) = \frac{n^2}{12}
\]

From (10) we have an interesting inequality, namely for all \(f_i\)

\[
(12) \quad V(f) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[|f_i - f_j|^2 - |f_i - f_j| + \frac{1}{6}\right] > 0
\]

This function is not convex even for \(n=2\), since \(f^{(1)} = (\frac{1}{2}, 0)\) and \(f^{(2)} = (-\frac{1}{2}, 0)\) yields \(V(f^0) = V(f^1) = \frac{1}{12} + \frac{1}{12} - \frac{1}{12} = \frac{1}{12}\) but
There appears to be no obvious direct way to establish that \( V(f) \geq 0 \) for all \( 0 \leq f_i \leq 1 \). Our development shows \( V(f) \) to be a variance and this, of course, constitutes an indirect proof.

We can replace (12) by a convex realization: Assume \( f_i > f_{i+1} \) for all \( i \), then the problem of finding \( \text{Min } V(f) \) can be rewritten:

\[
(13) \text{Find Min } [V(f)] = \sum_{1<j} (f_i - f_j)^2 + \frac{n^2}{12} - \left[ (n-1)f_1 + (n-3)f_2 + (n-5)f_3 + \ldots + (n-2k+1)f_k + \ldots - (n-1)f_n \right]
\]

subject to

\[
(14) f_1 > f_2 \ldots > f_n
\]

\[
(15) 0 \leq f_i \leq 1
\]

Formally (13), (14), (15), is a positive definite quadratic program. Fortunately, as we shall see this can be solved by classical calculus by ignoring inequalities (14) and (15).

**Theorem:** Equally spaced \( f_i = (n-i)/n \), \( i = 1, \ldots, n \) yields

\[
\text{Min } V(f) = \frac{1}{12} \text{ independent of } n, \text{ i.e., the variance of the sum in this case is minimum and is the same as the variance of the individual terms forming the sum.}
\]

**Proof:** Setting partials = 0 in (13) yields:
\[
\begin{align*}
2(n-1)f_1 & - 2f_2 \ldots - 2f_n = (n-1) \\
- 2f_1 + 2(n-1)f_2 & - 2f_n = (n-3) \\
- 2f_1 - 2f_2 \ldots 2(n-1)f_n 1 & - 2f_n = -(n-3) \\
- 2f_1 - 2f_2 & 2(n-1)f_n = -(n-1)
\end{align*}
\]

Adding shows the equations to be dependent. Hence we may drop the last equation as redundant. Moreover, we can always translate the \( f_i \) so that the smallest \( f_i \), namely \( f_n = 0 \)

Re-adding yields:

\[2f_1 + 2f_2 + \ldots 2f_n + 0 = (n-1) \quad , \quad f_n = 0.\]

Adding this last equation to each of the others gives

\[2nf_i = (n - 2i + 1) + (n - 1) = 2(n - i)\]

(17) \[f_i = (n - i)/n\]

Evidently the conditions \( 0 \leq f_i \leq 1 \) and \( f_i > f_{i+1} \) are (by good luck) also satisfied so that (17) yields the minimum, namely

\[\text{Min } V(f) = \frac{n^2}{12} - \frac{1}{2} \sum_{i=1}^{n} (n-2i+1)f_i = \frac{1}{12}.\]