THE MAXIMUM AND MINIMUM OF A POSITIVE DEFINITE
QUADRATIC POLYNOMIAL ON A SPHERE ARE
CONVEX FUNCTIONS OF THE RADIUS

BY

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Abstract

It is proved that in euclidean n-space the maximum $M(p)$ and minimum $m(p)$ of a fixed positive definite quadratic polynomial $Q$ on spheres with fixed center are both convex functions of the radius $p$ of the sphere. In the proof, which uses elementary calculus and a result of Forsythe and Golub, $m''(p)$ and $M''(p)$ are shown to exist and lie in the interval $[2\lambda_1, 2\lambda_n]$, where $\lambda_1$ are the eigenvalues of the quadratic form of $Q$. Hence $m''(p) > 0$ and $M''(p) > 0$. 
Summary

Let $A$ be a given symmetric, nonsingular matrix of real elements and order $n$. Let $b$ be a given column vector of $n$ real elements. For each real column $n$-vector $x$, the nonhomogeneous quadratic polynomial

$$ Q(x) = (x-b)^T A (x-b) $$

($T$ denotes transpose) is a real number. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the (necessarily) real eigenvalues of $A$. Let $m(p)$ be the minimum of $Q(x)$ on the sphere $S = \{x: x^T x = \rho \}$, and let $M(p)$ be the maximum of $Q(x)$ on $S$. M. J. D. Powell asked the author whether $m(p)$ is a convex function of $\rho$ when $A$ is positive definite. An affirmative answer is given by the theorem:

(1) **Theorem.** If $A$ is positive definite i.e., if $0 < \lambda_1$, then both $m(p)$ and $M(p)$ are convex functions of $\rho$, for all $\rho > 0$.

Theorem (1) will follow from the following result:

(2) **Theorem.** Let $A$ be any nonsingular matrix. Then for $\rho > 0$, the second derivatives $m''(\rho)$ and $M''(\rho)$ both exist, and

$$ m''(\rho) \geq 2\lambda_1 \quad \text{and} \quad M''(\rho) \geq 2\lambda_1. $$

Equality occurs in (3) if and only if $Ab = \lambda_1 b$. Moreover,

$$ m''(\rho) \leq 2\lambda_n \quad \text{and} \quad M''(\rho) \leq 2\lambda_n, $$

and equality occurs in (4) if and only if $Ab = \lambda_n b$. 

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Review of Previous Work

The proof of Theorem (2) is based on techniques developed in Forsythe and Golub [1], which dealt only with the case $\rho = 1$. The relevant results of [1] are now summarized and extended to general $\rho$.

Let $\{u_1, \ldots, u_n\}$ be an orthonormal real set of eigenvectors of $A$, with $Au_i = \lambda_i u_i$ ($i = 1, \ldots, n$). Let $b = \sum b_i u_i$. For any vector $x$ in $S_{\rho}$ at which $Q(x)$ is stationary with respect to $S_{\rho}$, there is a real number $\lambda$ with

\begin{equation}
A(x-b) = \lambda x
\end{equation}

\begin{equation}
x^T x = \rho^2
\end{equation}

Letting $x = \sum x_i u_i$, we find from (5) that

\begin{equation}
x_i = \frac{x_i b_i}{\lambda_i - \lambda}
\end{equation}

so that (6) becomes

\begin{equation}
g(\lambda) = \sum_{i=1}^{n} \frac{\lambda_i^2 b_i^2}{(\lambda_i - \lambda)^2} = \rho^2
\end{equation}

For each given value of $\rho > 0$, equation (8) determines from 2 to $2n$ real values of $\lambda$. For each $\lambda$ so determined, equation (5) determines one or more vectors $x_\lambda$ (if all $b_i \neq 0$, then $x_\lambda$ is unique). For any $x_\lambda$, we have

\begin{equation}
Q(x_\lambda) = f(\lambda),
\end{equation}

where
\[ f(h) = \lambda^2 \sum_{i=1}^{n} \frac{\lambda_i b_i^2}{(\lambda_i - \lambda)^2} \]

Now \( Q(x) \) is stationary with respect to \( S \) at any \( x^\lambda \). For given \( \rho \), let \( \Lambda_L = \Lambda_L(\rho) \) and \( \Lambda_R = \Lambda_R(\rho) \) be the smallest resp. largest values of \( \lambda \) satisfying equation (8). Theorem (4.1) of [1] states that \( f(\Lambda_L) \) and \( f(\Lambda_R) \) are the minimum resp. maximum values of \( Q(x) \) on \( S \).

Much of [1] was devoted to the singular cases where some \( b_i = 0 \).

For the present investigation, where we are interested only in the values of \( Q(x) \), we simply omit from the sums (8) and (10) all terms with \( b_i = 0 \), and reduce \( n \), if necessary. Having done that, it is then clear from (8) that, for any \( \rho \),

\[ \Lambda_L < \lambda_1 \quad \text{and} \quad \lambda_n < \Lambda_R. \]

This concludes the necessary summary of [1].

As a digression, the author notes that the main theorems (2.7) and (4.1) of [1] were proved in [1] by studying \( f(\lambda) \) and \( g(\lambda) \) for complex values of \( \lambda \). In late 1965, Professor W. Kahan [unpublished] showed us how to prove those theorems more simply, using only real values of \( \lambda \).

**Proof of Theorem (2).**

With the above apparatus our problem is reduced to an exercise in the differential calculus. For each \( \rho > 0 \) we determine a unique Lagrange multiplier \( \lambda = \lambda(\rho) \) from (8) -- either the minimal \( \Lambda_L \) or maximal \( \Lambda_R \).

For ease of exposition, suppose \( \lambda(\rho) = \Lambda_L \). Then the function

\[ m(\rho) = f(\lambda(\rho)) \]
is determined from (10). Since \( f(\lambda) \) and \( g(A) \) are analytic for \( \lambda < \lambda_1 \), the function \( m(p) \) has derivatives of all order. We shall determine \( m''(p) \) by calculus. To simplify some expressions, we introduce the abbreviations

\[
(13) \quad \alpha_p = \sum_{i=1}^{n} \frac{\lambda_i^2 b_i^2}{(\lambda_1 - \lambda)^i} \quad (p = 2, 3, 4).
\]

Differentiating (10) and simplifying, we find:

\[
(14) \quad \frac{df}{d\lambda} = 2\lambda \alpha_3 ;
\]

\[
(15) \quad \frac{d^2 f}{d\lambda^2} = 2\alpha_3 + 6\lambda \alpha_4 .
\]

Now equation (8) states that, when \( \lambda = \lambda(p) \),

\[
(16) \quad \alpha_2 = \rho^2 .
\]

Differentiating (8) twice with respect to \( \rho \) yields

\[
(17) \quad \frac{d\lambda}{d\rho} \alpha_3 = \rho ;
\]

\[
(18) \quad \frac{d^2 \lambda}{d\rho^2} \alpha_3 + \frac{3}{\alpha_3} \left( \frac{d\lambda}{d\rho} \right)^2 \alpha_4 = 1 .
\]

Solving (17) and (18) in turn, we find

\[
(19) \quad \frac{d\lambda}{d\rho} = \frac{\rho}{\alpha_3} ;
\]

\[
(20) \quad \frac{d^2 \lambda}{d\rho^2} = \frac{1}{\alpha_3} - \frac{3\rho^2 \alpha_4}{\alpha_3^2} .
\]
Now, by the chain rule,

\[
\frac{d^2m}{d\rho^2} = \frac{df}{d\lambda} \cdot \frac{d\lambda}{d\rho},
\]

and

\[
\frac{d^2m}{d\rho^2} = \frac{d^2f}{d\lambda^2} \left( \frac{d\lambda}{d\rho} \right)^2 + \frac{df}{d\lambda} \cdot \frac{d^2\lambda}{d\rho^2}.
\]

We now substitute into (21) the expressions (14), (15), (19), and (20).

We find that

\[
\frac{1}{2} m''(\rho) = \frac{\alpha_3}{3} \left( 2\alpha_3 + 6\lambda \alpha_4 \right) \frac{\rho^2}{\lambda^2} + 2\lambda \alpha_3 \left( \frac{1}{\lambda_3} - \frac{3\rho^2 \alpha_4}{\lambda_3^3} \right).
\]

Hence

\[
\frac{1}{2} m''(\rho) = \lambda + \frac{\rho^2}{\lambda_3} \left( \lambda \alpha_3 + \alpha_2 \right), \quad \text{by (16)}.
\]

Simplifying,

\[
\frac{1}{2} m''(\rho) = \frac{1}{\lambda_3} \sum_{i=1}^{n} \frac{\lambda_i^3 b_i^2}{(\lambda_i - \lambda)^3}, \quad \text{or}
\]

\[
\frac{1}{2} m''(\rho) = \frac{1}{\lambda_3} \sum_{i=1}^{n} \frac{\lambda_i^3 b_i^2}{(\lambda_i - \lambda)^3}.
\]

Formula (23) is the end of our calculus exercise. In it, \( \lambda \) is determined from solving (8). Note by (11) that the factors \((A_i - A)^3\) all have the same sign for \( i = 1, 2, \ldots, n \), whether \( \lambda = \lambda_L \) or \( \lambda = \lambda_R \).

Hence \( \frac{1}{2} m''(\rho) \) is a weighted average with positive weights of the \( \{\lambda_i\} \).
It follows that $\frac{1}{2} m''(\rho) \geq \lambda_1$, with equality only when all $\lambda_i$ in (23) are equal to $\lambda_1$, i.e., if $b_i = 0$ for $\lambda_i > \lambda_1$. This proves (3), and (4) is proved analogously. This concludes the proof of Theorem (2).

It would be desirable to have a simple geometrical proof.

What if A is singular?

If A is singular, that is, if some $\lambda_i = 0$, the situation is somewhat more complicated, just as the case where some $\lambda_i b_i = 0$ is complicated in [1]. Theorem (2) fails to hold for semidefinite matrices, because $m''(\rho)$ may not exist for some $\rho$, as the following example shows:

(24) Example. For $n = 2$ let $Q(x) = (x_2 - 1)^2$; $x = (x_1, x_2)^T$.

Then

$$m(\rho) = \begin{cases} 1 - \rho & , \ 0 \leq \rho \leq 1, \\ 0 & , \ 1 \leq \rho < \infty, \end{cases}$$

so $m'(1)$ does not exist.

If $\lambda_1 = 0$, the Lagrange multiplier remains at $\lambda = 0$ for all sufficiently large $\rho$.

Theorem (1) can easily be extended to semidefinite matrices by continuity. We have

(25) Theorem. If A is positive semidefinite (i.e., if $0 \leq \lambda_1$), then both $m(p)$ and $M(p)$ are convex functions of $\rho$ for $\rho > 0$.

In proof, we note that $m(p)$ and $M(p)$ are continuous functions of the elements of A. If A is semidefinite, it can be approximated by a definite matrix $A_\varepsilon$, for which $m_\varepsilon$ and $M_\varepsilon$ are convex, with $||A - A_\varepsilon|| < \varepsilon$.

Letting $\varepsilon \to 0$, we find that $m = \lim m_\varepsilon$ and $M = \lim M_\varepsilon$ are convex.
Reference

[1] George E. Forsythe and Gene H. Golub, "On the stationary values of a second-degree polynomial on the unit sphere",