A Computational Group-Theoretic Approach to Steerable Functions

Patrick C. Teo
Department of Computer Science
Stanford University, Stanford, CA 94305
teo@white.Stanford.EDU

Yacov Hel-Or
NASA Ames Research Center
Moffett Field, CA 94035-1000
toky@vision.arc.nasa.gov

Abstract

We present a computational, group-theoretic approach to steerable functions. The approach is group-theoretic in that the treatment involves continuous transformation groups for which elementary Lie group theory may be applied. The approach is computational in that the theory is constructive and leads directly to a procedural implementation. For functions that are steerable with n basis functions under a k-parameter group, the procedure is efficient in that at most $nk + 1$ iterations of the procedure are needed to compute all the basis functions. Furthermore, the procedure is guaranteed to return the minimum number of basis functions. If the function is not steerable, a numerical implementation of the procedure could be used to compute basis functions that approximately steer the function over a range of parameters. Examples of both applications are described.

1 Introduction

Steerable functions have been widely used in image processing [3, 13, 12, 11], computer vision [5, 8, 16, 7, 2, 15], and recently, even in computer graphics [4, 10]. Simply put, a function is steerable under some transformation when all transformed versions of the function can be expressed as a linear combination of a fixed, finite set of basis functions, the weights of the linear combination depending solely on the transform parameters.

The importance of steerable functions stems from the property of superposition of linear systems. Hence, any linear operation applied to a transformed version of a steerable function can be expressed as a linear combination of the operation applied separately to the basis functions. The main advantage of this property is that the linear operations can be applied to the basis function once and off-line. In image processing, steerable functions have been used as filter kernels. Because convolution is a linear operation, the filter output of a transformed version of the filter kernel is obtained by linearly combining the filter outputs of its associated basis filters.

Freeman and Adelson presented functions steerable with respect to rotation using derivatives of a Gaussian as the basis set [3]. An extension of this technique to translation and scaling was shown by Simoncelli et al. [13]. Indeed, steerable filters are the most popular application of steerable functions; thus, the examples in this paper will pertain mainly to steerable filters.

Many of the commonly encountered families of transformations on images form continuous groups. Examples of these continuous groups include: image translation, rotation and scaling. For these families of transformations, Lie group-theoretic tools have been used to analyze the property of steerability and to categorize the classes of functions which are steerable under different transformation groups. In spite of this, given an arbitrary function, there are several ways of computing a set of basis functions to steer an arbitrary function. For example, Perona [11] suggests sampling the space of transform parameters by constructing replicas of the common kernel, each of which is transformed according to a particular choice of parameters. The n-largest principal components of this set are then used to steer the given filter. Alternatively, Simoncelli *et. al.* [12], Michaelis *et. al.*[9] and Hel-Or *et. al.*[6] first approximates the function to be steered by a set of steerable functions, and then steers the approximated function by analytically steering the set of steerable functions.

Perona's method is optimal in the sense that it computes the best set of basis functions that minimize a least-squares error criterion. However, for groups involving even a modest number of parameters, brute force sampling of the space can make the method computationally inefficient. The approximation method employed by Simoncelli, Michaelis, Hel-Or and their co-authors does not suffer from this problem.

While the scheme avoids explicit enumeration, it has some disadvantages. First, it requires that the steerable function spaces be identified in advance. Second, a fair approximation of a compactly supported function on these spaces may require a large number of basis functions as these basis functions are typically not compactly supported. Hel-Or *et. al.* [6] proposes a solution to this problem by limiting the domain of approximation for functions that are to be steered only over a restricted range of parameters.

In this paper, we present a computational, group-theoretic approach to steerable functions. The approach is group-theoretic in that the treatment involves continuous transformation groups for which elementary Lie group theory may be applied. The approach is computational in that the theory is constructive and leads directly to a procedural implementation. We describe a procedure to compute a set of basis functions to steer an arbitrary function. If the function is steerable with n basis functions under a k-parameter group, then the procedure is guaranteed to terminate in at most $nk + 1$ iterations. We also present results from applying this technique to: (1) analytically computing a set of basis functions to steer an arbitrary polynomial under any subgroup of the two-dimensional affine group, and (2) numerically computing a set of basis functions to approximately steer any given function (from a sampled version of it).

The rest of the paper is organized as follows. Before presenting the approach, a brief review of Lie transformation groups is given in Section 2. Following that, the approach is described in detail; first, in Section 3 for one-parameter transformation groups, and then, in Section 4 for multiple-parameter groups. The application of the approach to steering polynomials and to a numerical implementation are given next in Section 5. The paper is concluded in Section 6 with some discussion.

2 Background on Lie Groups

Lie groups are often encountered as families of transformations acting on a function. In this paper, we consider, primarily, the families of transformation groups acting on real-valued, two-dimensional functions. We assume that these functions are non-zero only within a bounded region and denote them by $f(x, y) : \mathbf{R}^2 \mapsto \mathbf{R}$. We describe each family of transformations by *operators* $\{g(\boldsymbol{\tau})\}$, where $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_k) \in \mathbf{R}^k$ are parameters of the transformation. For example, consider the family of one-dimensional translations of a function in the x-direction:

$$\hat{f}(\hat{x}, \hat{y}) = g_{t_x}(\tau)\, f(x, y) = f(x - \tau, y)$$

where τ denotes the amount of translation. In words, the operator $g_{t_x}(\tau)$ acts on the original function $f(x,y)$ to yield a new translated function $\hat{f}(\hat{x},\hat{y}) = f(x-\tau, y)$.

A family of transformations $\{g(\boldsymbol{\tau})\}$ parameterized by $\boldsymbol{\tau}$ over some predefined range is a Lie group if: (1) it satisfies the group conditions of closure under composition, associativity, inverse and the existence of an identity, and (2) the maps for inverse and composition are smooth. Thus, the family of translations forms a Lie group: First, every translation operator $g_{t_x}(\tau)$ has an inverse, namely, $g_{t_x}(\iota(\tau))$ where $\iota(\tau) = -\tau$. Since $\iota(0) = 0$, $g_{t_x}(0)$ is also the identity operator. Second, composition of two operators can be described by a third operator which also belongs to the same family; i.e. $g_{t_x}(\tau_a)g_{t_x}(\tau_b) = g_{t_x}(\rho(\tau_a,\tau_b))$ where $\rho(\tau_a,\tau_b) = \tau_a + \tau_b$. In addition, composition is associative, that is to say, $g_{t_x}(\tau_a)(g_{t_x}(\tau_b)g_{t_x}(\tau_c)) = (g_{t_x}(\tau_a)g_{t_x}(\tau_b))\, g_{t_x}(\tau_c)$; as such, $\rho(\tau_a, \rho(\tau_b,\tau_c)) = \rho(\rho(\tau_a,\tau_b),\tau_c)$. Finally, both the inverse map, $\iota(\tau)$, and the composition map, $\rho(\tau_a,\tau_b)$ are smooth. The dimension of the parameter space of a Lie transformation group may be different from the dimension of the function space upon which it acts. Here, the family of translations in the x-direction forms a one-parameter Lie group ($\tau \in \mathbf{R}$) while the space upon which it acts is two-dimensional ($(x,y) \in \mathbf{R}^2$).

Another familiar family of transformations that is also a Lie group is the group of rotations in the plane $g_r(\tau)$ such that $\hat{f}(\hat{x},\hat{y}) = g_r(\tau)\, f(x,y) = f(x\cos\tau - y\sin\tau, x\sin\tau + y\cos\tau)$. It is straightforward to check that the necessary conditions, verified in the previous example, are also satisfied here. The family of transformations defined by $g_b(\tau)\, f(x,y) = e^{\tau} f(x,y)$ scales the amplitude of the function uniformly and also forms a Lie group. However, unlike translations and rotations in the plane, $g_b(\tau)$ acts on the function itself and not on the coordinates of the function. In this paper, we will deal exclusively with Lie transformation groups; i.e. Lie groups that operate solely on the coordinates of the image.

Lie groups are rich in structure and many properties of the group can be discerned by studying the properties of infinitesimal actions of the group. In the following, infinitesimal actions of a group are defined and elaborated. We consider first one-parameter groups and then extend our explanation to multi-parameter groups.

One-parameter Groups Given a one-parameter transformation group parameterized by τ, the infinitesimal transformation of an function $f(x,y)$ about the identity ($\tau = 0$) is defined using Leibnitz's chain rule:

$$\frac{d}{d\tau}\left(g(\tau)\, f\right)\bigg|_{\tau=0} = \frac{d\hat{f}}{d\tau}\bigg|_{\tau=0} = \left(\frac{\partial x}{\partial \tau}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \tau}\frac{\partial}{\partial y} + \frac{\partial}{\partial \tau}\right)\bigg|_{\tau=0} \hat{f}.$$

The differential operator on the right hand side of the equation is called the (infinitesimal) *generator* of the transformation and is denoted by L; i.e.

$$L = \left(\frac{\partial x}{\partial \tau} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \tau} \frac{\partial}{\partial y} + \frac{\partial}{\partial \tau} \right) \bigg|_{\tau=0} \tag{1}$$

The set of elements $\mathcal{G} = \{\tau L \mid \tau \in \mathbf{R}\}$ forms a one-dimensional vector space called the *tangent space* of the group where L can be thought of as a one-dimensional basis vector. There is a strong connection between the tangent space and the Lie group from which it was derived. Namely, each element $g(\tau)$ of the group can be generated by an element in the tangent space, $\tau L \in \mathcal{G}$, via the exponential map:[1]

$$g(\tau)\, f(x, y) = e^{\tau L}\, f(x, y)\ . \tag{2}$$

The notation $e^{\tau L}$ represents the series expansion $e^{\tau L} = I + \tau L + \frac{1}{2!}\tau^2 L^2 + \cdots$, which is an infinite sum of differential operators [1]. This is a rather surprising result since the operator $g(\tau)$ can transform the function in highly nonlinear ways while $\mathcal{G}$ is simply a linear vector space.

Recall the group of translations in the x-direction presented earlier. The derivative of the transformation about the identity is

$$\frac{d}{d\tau}(g_{t_x}(\tau) f)\bigg|_{\tau=0} = \frac{\partial(x-\tau)}{\partial \tau}\bigg|_{\tau=0} \frac{\partial}{\partial x}\, f = -\frac{\partial}{\partial x}\, f$$

and hence its generator is $L_{t_x} = -\frac{\partial}{\partial x}$. Using the exponential map suggested in Equation 2, we find that

$$\begin{aligned} g_{t_x}(\tau)\, f &= e^{\tau L_{t_x}}\, f \\ &= (1 - \tau \tfrac{\partial}{\partial x} + \tfrac{1}{2!}\tau^2 \tfrac{\partial^2}{\partial x^2} - \cdots)\, f \\ &= f - \tau \tfrac{\partial f}{\partial x} + \tfrac{1}{2!}\tau^2 \tfrac{\partial^2 f}{\partial x^2} - \cdots \end{aligned}$$

which is exactly the Taylor expansion of $f(x - \tau, y)$ about $\tau = 0$. Further examples of one-parameter groups and their generators are given in Table 1.

Multi-parameter Groups The situation with multiple-parameter Lie groups is analogous. The generators of a multi-parameter group are the set of differential operators $\{L_i \mid i = 1 \ldots k\}$ corresponding to derivatives of the transformation at the identity

[1] To be precise, this is only true for group elements sufficiently close to the identity element so that their Taylor expansions converge, and for elements within the connected component containing the identity. In this paper, we consider only transformation groups with one connected component and for which convergence also holds.

Group	Operator	Generator
x-translation	$g_{t_x}(\tau)\ f(x,y) = f(x-\tau, y)$	$L_{t_x} = -\frac{\partial}{\partial x}$
x-scaling	$g_{s_x}(\tau)\ f(x,y) = f(e^{-\tau}x, y)$	$L_{s_x} = -x\frac{\partial}{\partial x}$
x-projective	$g_{p_x}(\tau)\ f(x,y) = f(x/(1+\tau x), y)$	$L_{p_x} = -x^2\frac{\partial}{\partial x}$
y-translation	$g_{t_y}(\tau)\ f(x,y) = f(x, y-\tau)$	$L_{t_y} = -\frac{\partial}{\partial y}$
y-scaling	$g_{s_y}(\tau)\ f(x,y) = f(x, e^{-\tau}y)$	$L_{s_y} = -y\frac{\partial}{\partial y}$
y-projective	$g_{p_y}(\tau)\ f(x,y) = f(x, y/(1+\tau y))$	$L_{p_y} = -y^2\frac{\partial}{\partial y}$
Rotation	$g_r(\tau)\ f(x,y) = f(x\cos\tau - y\sin\tau,$	
	$x\sin\tau + y\cos\tau) = f(r, \theta-\tau)$	$L_r = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = -\frac{\partial}{\partial\theta}$
Uniform scaling	$g_s(\tau)\ f(x,y) = f(e^{-\tau}x, e^{-\tau}y) = f(e^{\tau}r, \theta)$	$L_s = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} = -\frac{\partial}{\partial r}$

Table 1: Several examples of continuous transformation groups and their generators. Multi-parameter groups can be constructed by combining several of these groups.

with respect to each parameter τ_i in turn; i.e.

$$\left.\frac{d\hat{f}}{d\tau_i}\right|_{\boldsymbol{\tau}=\mathbf{0}} = L_i\ \hat{f}$$

where

$$L_i = \left.\left(\frac{\partial x}{\partial \tau_i}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \tau_i}\frac{\partial}{\partial y} + \frac{\partial}{\partial \tau_i}\right)\right|_{\boldsymbol{\tau}=\mathbf{0}}.$$

The k generators provide a basis for the k-dimensional tangent space $\mathcal{G} = \{\tau_1 L_1 + \cdots + \tau_k L_k | \boldsymbol{\tau} \in \mathbf{R}^k\}$.[2] As before, there is a correspondence between a k-parameter Lie group and its k-dimensional tangent space in the form of the exponential map:

$$g(\boldsymbol{\tau})\ f(x,y) = (\prod_{i=1}^{k} e^{\tau_i L_i})\ f(x,y) = e^{\tau_1 L_1} \cdots e^{\tau_k L_k}\ f(x,y). \tag{3}$$

Although the exponential map provides a correspondence between every operator in the Lie group and every element in its tangent space, the parameterization of the group generated by the exponential map may be different from that of the original group. The order of the individual exponentials $e^{\tau_i L_i}$ in Equation 3 is arbitrary such that different orderings give rise to different group parameterizations. Hence, the exponential map generates a group similar to the original group up to a change of parameterization. For example, consider the two parameterizations of the two-parameter affine group acting

[2] Loosely speaking, the linear independence of the k generators is assured if the k-parameter group from which it was derived cannot be replaced by another with fewer parameters [1].

solely on the x coordinate:

$$\begin{aligned} g_1(\tau_1, \tau_2)\, f(x,y) &= f(e^{\tau_1} x - \tau_2, y), \\ g_2(\tau_1, \tau_2)\, f(x,y) &= f(e^{\tau_1}(x - \tau_2), y). \end{aligned}$$

Both yield the same generators:

$$L_{\tau_1} = x\frac{\partial}{\partial x}, \qquad L_{\tau_2} = -\frac{\partial}{\partial x};$$

thus, both have the same exponential map. However, this is not a problem as we are often interested in the group of transformations and not the particular parameterization of it. Furthermore, we can easily reparameterize the generated group using the original parameterization.

With multi-parameter groups, if we vary a single parameter τ_i and keep the others fixed, we get a one-parameter group of transformations $\{g_i(\tau_i)\}$ that is a *subgroup* of the original k-parameter group. Hence, by varying each of the k different parameters separately, we can construct k different one-parameter subgroups. When every element from one subgroup commutes with every element from a second subgroup, i.e. $g_i(\tau_i)\, g_j(\tau_j) = g_j(\tau_j)\, g_i(\tau_i)$ for all τ_i, τ_j, the two subgroups are said to commute with each other. Two subgroups commute if and only if their Lie bracket vanishes; i.e. $[L_i, L_j] \doteq L_i L_j - L_j L_i = 0$ [1]. When two subgroups commute, exponentiating their respective generators can be done in either order; i.e. $e^{\tau_i L_i} e^{\tau_j L_j} = e^{\tau_j L_j} e^{\tau_i L_i} = e^{\tau_i L_i + \tau_j L_j}$. This is not true for non-commuting subgroups. A multi-parameter group for which all its one-parameter subgroups commute is called an *Abelian* group. The two-parameter group of the previous example is not Abelian since

$$[L_{\tau_1}, L_{\tau_2}] = -x\frac{\partial}{\partial x}(\frac{\partial}{\partial x}) + \frac{\partial}{\partial x}(x\frac{\partial}{\partial x}) = \frac{\partial}{\partial x} \neq 0.$$

Hence, as demonstrated earlier, $e^{\tau_1 L_1} e^{\tau_2 L_2} \neq e^{\tau_2 L_2} e^{\tau_1 L_1} \neq e^{\tau_1 L_1 + \tau_2 L_2}$.

On the other hand, for the one-parameter transformation groups listed in Table 1, the pairs, $\{g_{t_x}, g_{t_y}\}, \{g_{t_x}, g_{s_y}\}, \{g_{t_y}, g_{s_x}\}, \{g_{s_x}, g_{s_y}\}, \{g_r, g_s\}$, are commutative.

3 Generator Chains

In this section, we describe our approach to steerable functions in the context of one-parameter transformation groups; the treatment of multi-parameter transformation groups is deferred to the next section. We begin by formalizing the notion of a steerable function with the following definition.

Definition 1 (Steerability) *: A function* $f(x,y) : \mathbf{R}^2 \mapsto \mathbf{R}$ *is* **steerable** *under a Lie transformation group G if any transformation* $g(\boldsymbol{\tau}) \in G$ *applied to f can be written as a linear combination of a fixed, finite set of* basis functions $\{\phi_i(x,y)\}$*:*

$$g(\boldsymbol{\tau})\, f(x,y) = \sum_{i=1}^{n} \alpha_i(\boldsymbol{\tau})\, \phi_i(x,y) = \boldsymbol{\alpha}^T(\boldsymbol{\tau})\Phi(x,y)$$

where $\boldsymbol{\alpha}^T = (\alpha_1, \cdots, \alpha_n)$ *and* $\Phi(x,y) = (\phi_1, \cdots, \phi_n)^T$*.*

The functions α_i are known as the *steering functions* of f associated with the basis $\{\phi_i\}$ and depend solely on the transformation parameters. Without loss of generality, we assume that n is the minimum number of basis functions required and these basis functions are linearly independent.

The set of basis functions required to steer a given function is not unique; any (non-singular) linear transformation of the set of basis functions could also be used. Furthermore, if n is the minimum number of basis functions required to steer a function f, then any steerable basis of f will require only n basis functions as well. If a particular steerable basis contains $m > n$ basis functions, then $m - n$ of them are necessarily linearly dependent on the rest.

Theorem 1 (minimality of basis functions) *: Let* $\Phi = (\phi_1, \cdots, \phi_n)^T$ *be the minimum set of independent basis functions required to steer a function f under a Lie transformation group G. Then, any other steerable basis* $\Omega = (\omega_1, \cdots, \omega_m)^T$ *of f with respect to G has exactly n linearly independent functions.*

Proof 1:

Assume that m is the minimum number of linearly independent functions in Ω to steer f. Therefore, it is possible to find m transformed replicas of f that are linearly independent (otherwise m is not minimal):

$$\begin{pmatrix} g(\boldsymbol{\tau}^1)f \\ \vdots \\ g(\boldsymbol{\tau}^m)f \end{pmatrix} = \begin{bmatrix} \boldsymbol{\beta}^T(\boldsymbol{\tau}^1) \\ \vdots \\ \boldsymbol{\beta}^T(\boldsymbol{\tau}^m) \end{bmatrix} \Omega \doteq \boldsymbol{B}\Omega$$

where $\boldsymbol{\beta}^T(\boldsymbol{\tau})$ is composed of the steering functions associated with Ω and $\boldsymbol{\tau}^1 \ldots \boldsymbol{\tau}^m$ are particular choices of steering parameters. Since the m transformed replicas are linearly independent, $\boldsymbol{B}$ is a non-singular matrix. These m transformed replicas of f can also be constructed using the n basis functions of Φ:

$$\begin{pmatrix} g(\boldsymbol{\tau}^1)f \\ \vdots \\ g(\boldsymbol{\tau}^m)f \end{pmatrix} = \begin{bmatrix} \boldsymbol{\alpha}^T(\boldsymbol{\tau}^1) \\ \vdots \\ \boldsymbol{\alpha}^T(\boldsymbol{\tau}^m) \end{bmatrix} \Phi \doteq \boldsymbol{A}\Phi = \boldsymbol{B}\Omega \ .$$

Since $\boldsymbol{B}$ is invertible it is possible to express Ω as a linear combination of Φ:

$$\Omega = \boldsymbol{B}^{-1}\boldsymbol{A}\Phi.$$

Now, if Ω includes $m > n$ functions, it is obvious from the above equation that $m - n$ of them are linearly dependent. This contradicts the minimality assumption of m. On the other hand, if $m < n$, then n is not minimal. Thus, it must be true that $m = n$ and all the n functions in Ω are linearly independent. $\square$

In the following, we describe a method of constructing a basis for a given steerable function f under a transformation G. From Theorem 1, this basis can be related to any other steerable basis of f via a linear transformation.

Associated with each one-parameter transformation group G is its generator L. As shown in Section 2, the generator L is a differential operator corresponding to an infinitesimal transformation about the identity. Applying L to a function f results in a new function Lf; likewise, applying L a second time to the previous result yields another function which we denote by $L^2 f = L(Lf)$. Alternatively, we could also regard L^2 (or L^j, $j \geq 0$) as a new differential operator that is applied to f. The set of all such differential operators is collected into a sequence in the following definition.

Definition 2 (Generator Chain) *: A* generator chain $\mathcal{C}(L)$ *is an ordered sequence of differential operators corresponding to repeated applications of the given generator* L*; i.e.,*

$$\mathcal{C}(L) = (I,\ L,\ L^2,\ L^3,\ \ldots)$$

where I *corresponds to zero applications of the generator.*

The result of applying $\mathcal{C}(L)$ to a function f is defined to be the ordered sequence of functions:

$$\mathcal{C}(L)\ f = (f,\ Lf,\ L^2 f,\ L^3 f,\ \ldots).$$

Using the exponential map given in Equation 2, the series formed by summing all the functions in the sequence is exactly the same as transforming f by an element $g(\tau) \in G$:

$$g(\tau)\ f = \exp(\tau L)\ f = \sum_{i=0}^{\infty} \frac{\tau^i}{i!} \left(L^i f\right). \tag{4}$$

Thus, the set of functions $\mathcal{C}(L)\, f$ provides a basis with which f can be steered. From Theorem 1 it follows that this basis is redundant if only n functions are required to steer f; only n of the functions in $\mathcal{C}(L)\, f$ are linearly independent. It turns out that these n functions are necessarily the *first* n functions of the chain. The minimality of the generator chain is formalized in the following theorem.

Theorem 2 (Minimality of Generator Chain) *: Let f be a steerable function under a one-parameter Lie transformation group G such that transformations of f by any element $g \in G$ can be written as a linear combination of (no less than) n basis functions. Let L denote the generator of G. The application of the generator chain $\mathcal{C}(L)$ to f results in an ordered sequence of functions such that the elements $i > n$ of the sequence, corresponding to $L^{(i-1)}f$, are linearly dependent on the first n elements. Furthermore, the first n functions of the sequence are linearly independent.*

Proof 2 : Let the $(m+1)^{\text{th}}$ function in the sequence be the first linearly dependent function. That is, $L^m f$ can be written as a linear combination of the first m linearly independent functions. As a result, all subsequent functions in the sequence $L^j f$ where $j > m$ are necessarily linearly dependent on the first m functions as well. This can be proven by induction where $j = m+1$ is the base case. Let $L^m f = \sum_{i=1}^m a_i L^{(i-1)} f$. Then,

$$\begin{aligned}
L^{m+1} f &= L(L^m f) \\
&= L(\textstyle\sum_{i=1}^m a_i L^{(i-1)} f) \\
&= \textstyle\sum_{i=1}^m a_i L^i f \\
&= a_m L^m f + \textstyle\sum_{i=1}^{m-1} a_i L^i f \\
&= a_m (\textstyle\sum_{i=1}^m a_i L^{(i-1)} f) + \textstyle\sum_{i=1}^{m-1} a_i L^i f \\
&= a_m a_1 f + \textstyle\sum_{i=1}^{m-1} (a_m a_{i+1} + a_i) L^i f.
\end{aligned}$$

Thus, $L^{m+1} f$ can also be expressed as a linear combination of the first m functions. The proof of the inductive case is similar. As a result, Equation 4 implies that transformations of f can be written as a linear combination of the first m functions in $\mathcal{C}(L)\, f$. Because f is steerable with n basis functions, it follows from Theorem 1 that m must equal n. □

This theorem suggests the following procedure to compute a set of basis functions to steer an arbitrary function f. The generator L is applied to f repeatedly and each time, the linear dependence of the new function upon the previously computed functions is checked. If it is linearly dependent, then the set of all functions computed prior to this one is sufficient to steer f. It the function f is steerable with n basis functions, then the procedure will terminate after $n+1$ iterations. Figure 1 describes the procedure in pseudo-code. The procedure is applied to the following two examples.

```
/* Compute the basis functions needed to steer f */
/* under a one-parameter group. */
next_f = f;
basis_set = {};
while (next_f not linearly dependent on basis_set) {
      basis_set = basis_set ∪ {next_f}
      next_f = L next_f;
}
return( basis_set );
```

Figure 1: Procedure for computing the basis functions to steer an arbitrary function f under a one-parameter group.

Example 1 : Let $f(x, y) = -2x\ e^{-(x^2+y^2)}$ and G be the group of rotations in the plane: $g_r(\tau)\ f(x, y) = f(x\cos\tau - y\sin\tau, x\sin\tau + y\cos\tau)$. The generator of G is $L = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$, and

$$L^0 f = -2x\ e^{-(x^2+y^2)} = f, \qquad\qquad L^1 f = 2y\ e^{-(x^2+y^2)},$$

and $L^2 f = 2x\ e^{-(x^2+y^2)} = -f$. Therefore, f is steerable under G with two basis functions; i.e., the derivative of a Gaussian in any direction can be expressed as a linear combination of two functions.

Example 2 : Let $f(\theta) = (\cos\theta + 1)^2$ and G be the group of rotations: $g_r(\tau)\ f(\theta) = f(\theta - \tau)$. The generator of G is $L = -\frac{\partial}{\partial\theta}$, and

$$\begin{aligned}
L^0 f &= (\cos\theta + 1)^2, \\
L^1 f &= 2(\cos\theta + 1)\sin\theta = \sin 2\theta + 2\sin\theta, \\
L^2 f &= -2\cos 2\theta - 2\cos\theta, \\
L^3 f &= -4\sin 2\theta - 2\sin\theta, \\
L^4 f &= 8\cos 2\theta + 2\cos\theta,
\end{aligned}$$

and $L^5 f = 16\sin 2\theta + 2\sin\theta = -4L^1 f - 5L^3 f$. Therefore, f is steerable under G with the five basis functions, $\{L^0 f, \ldots, L^4 f\}$. The particular choice of basis functions is not unique; in this example, the function f is also steerable with the following five basis functions, $\{1, \sin\theta, \cos\theta, \sin 2\theta, \cos 2\theta\}$.

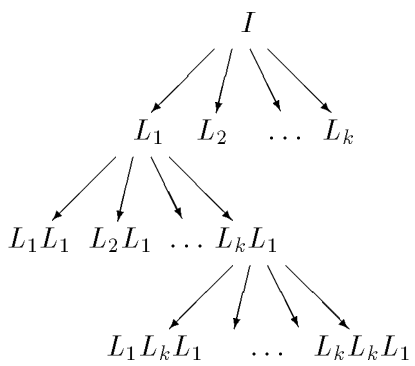
4 Generator Trees

In this section, we consider multi-parameter transformation groups. Let the set of differential operators, $\{L_1, \dots, L_k\}$, be the k generators of the k-parameter transformation group G. In the context of multi-parameter transformation groups, the generator chain is no longer a chain since more than one generator may be applied; instead, we have a tree of differential operators. Nodes in the tree correspond to all possible compositions of the generators.

Definition 3 (Generator Tree) *: A* generator tree $\mathcal{T}(L_1, \dots, L_k)$ *is a k-ary tree of differential operators corresponding to repeated applications of the generators $L_1, \dots, L_k$. Each node of the tree has k children, which correspond to applying each of the L_k different generators. Level l of the tree contains k^l nodes, each of which represents the different permutations of applying $L_1, \dots, L_k$ repeatedly for a total of l times.*

For example,

$\mathcal{T}(L_1, \dots, L_k) \quad =$



Similar to generator chains, applying $\mathcal{T}(L_1, \dots, L_k)$ to a function f results in a k-ary tree where each node corresponds to the function obtained by applying the generators to f. Furthermore, using the exponential map given in Equation 3, transforming f by an element $g(\tau_1, \dots, \tau_k) \in G$ can be calculated by a linear combination of functions in the tree $\mathcal{T}(L_1, \dots, L_k)\, f$:

$$g(\tau_1, \dots, \tau_k)\, f = e^{\tau_1 L_1} \cdots e^{\tau_k L_k}\, f = \left(\prod_{i=1}^{k} \sum_{l=0}^{\infty} \frac{\tau_i^l}{l!} L_i^l \right) f \tag{5}$$

Thus, the set of functions obtained by applying $\mathcal{T}(L_1, \dots, L_k)$ to a function f provides a basis with which to steer f. Similar to the case with generator chains, this basis is redundant if only n functions are required to steer f. It turns out that the n functions needed to steer f necessarily form a subtree of $\mathcal{T}(L_1, \dots, L_k)\, f$ with the same root. This property generalizes the minimality property of generator chains associated with one-parameter groups.

Theorem 3 (Minimality of Generator Tree) *: Let f be a steerable function under a k-parameter Lie transformation group G such that transformations of f by any element $g \in G$ can be written as a linear combination of (no less than) n basis functions. Let $L_1, \ldots, L_k$ denote the generators of G. The application of the generator tree $\mathcal{T}(L_1, \ldots, L_k)$ to f results in a k-ary tree of functions such that there exists a subtree $\mathcal{T}'(L_1, \ldots, L_k)\ f$ (with the same root) satisfying the following two conditions: (1) all functions within the subtree are linearly independent of one another, and (2) all functions in the original tree but not in the subtree are linearly dependent on functions within the subtree.*

Proof 3 : Let $\mathcal{T}'(L_1, \ldots, L_k)\ f$ be a subtree of $\mathcal{T}(L_1, \ldots, L_k)\ f$ (with the same root) such that: (1) all the functions within the subtree are linearly independent, and (2) all the functions that are immediate children of the subtree (as part of the original tree) are linearly dependent on the functions within the subtree. Then, all the descendents of the immediate children are also linearly dependent on the functions within the subtree. This can be proven in a way similar to that for generator chains in Theorem 2. The induction, in this case, is on subtrees. Also, from Theorem 1 it follows that the total number of linearly independent functions in the original tree is necessarily n since f is steerable. As a result of the former property, the size of the subtree must be n as well. $\square$

Unlike the situation with generator chains, this minimal subtree is not unique. That is, there may be two subtrees of the same size (and with the same root as the original tree) that could be used to steer f. However, since f is steerable, the functions in these two trees necessarily span the same space.

This theorem also suggests a procedure for computing the basis functions needed to steer an arbitrary function f. Each of the generators $\{L_1, \ldots, L_k\}$ is applied to f repeatedly. Each new function is then checked to determine if it is linearly dependent on all the previously computed functions. If it is linearly dependent, then one need not further apply any generators to this function for according to the theorem, doing so will only produce functions that are linearly dependent as well. If the function f is steerable with n basis functions under a k-parameter group, then the procedure will terminate after testing at most $nk + 1$ functions. The proof of this claim is given in Appendix A. Figure 2 describes the procedure in pseudo-code. The procedure is applied to the following two examples.

```
/* Compute the basis functions needed to steer f */
/* under a k-parameter group. */
next_set = { f };
basis_set = { f };
while (next_set is not empty) {
      next_set' = {};
      for each (next_f ∈ next_set) {
            for each (L ∈ {L_1, ..., L_k}) {
                  next_f' = L next_f;
                  if (next_f' not linearly dependent on basis_set) {
                     next_set' = next_set' ∪ {next_f'};
                     basis_set = basis_set ∪ {next_f'};
                  }
            }
      next_set = next_set';
      }
}
return( basis_set );
```

Figure 2: Procedure for computing the basis functions to steer an arbitrary function f under a k-parameter group.

Example 3 : Let $f(x) = x^2$ and G be the group of one-dimensional scaling and translations: $g(\tau_1, \tau_2)\, f(x) = f(e^{-\tau_1} x - \tau_2)$. The generators of G are $L_{\tau_1} = -x\frac{\partial}{\partial x}$ and $L_{\tau_2} = -\frac{\partial}{\partial x}$:

$$\mathcal{T}(L_{\tau_1}, L_{\tau_2})\, f \quad = \quad \begin{tikzcd}[column sep=small] x^2 \arrow[dr] & & \\ & -2x \arrow[dr] & \\ & & 2 \end{tikzcd} \quad .$$

Since $L_{\tau_1} f = -2x^2 = -2f$, the entire left subtree is linearly dependent. Now, $L_{\tau_2} f = -2x$ and $L_{\tau_2} L_{\tau_2} f = 2$ are linearly independent but $L_{\tau_1} L_{\tau_2} f = 2x = -L_{\tau_2} f$, $L_{\tau_2} L_{\tau_2} L_{\tau_2} f = 0$, and $L_{\tau_1} L_{\tau_2} L_{\tau_2} = 0$. Therefore, f is steerable under G with three basis functions.

Example 4 : Let $f(x, y) = \sin x \sin y$ and G be the group of translation along the x and y dimensions: $g(\tau_1, \tau_2)\, f(x) = f(x - \tau_1, y - \tau_2)$. The generators of G are $L_{\tau_1} = -\frac{\partial}{\partial x}$ and $L_{\tau 2} = -\frac{\partial}{\partial y}$:

$$\mathcal{T}(L_{\tau_1}, L_{\tau_2})\, f \quad = \quad \begin{tikzcd}[column sep=small] & \sin x \sin y \arrow[dl] \arrow[dr] & \\ -\cos x \sin y \arrow[dr] & & -\sin x \cos y \\ & \cos x \cos y & \end{tikzcd} \quad .$$

Since translation in the x and y dimensions are commutative, their generators commute as well; i.e., $L_{\tau_1} L_{\tau_2} = L_{\tau_2} L_{\tau_1}$. Thus, the left child of the node with $-\sin x \cos y$ is automatically pruned since it will be the same as the right child of the node with $\cos x \cos y$. Therefore, f is steerable under G with four basis functions.

5 Simulations

In this section, we present two applications of the theory described in the previous section. The first application is an implementation of the procedure in Figure 2 for steering polynomials. The second application is a numerical implementation of the same procedure for approximately steering any sampled function.

5.1 Steering Polynomials

The procedure in Figure 2 was implemented in MATLAB to automatically determine the basis functions needed to steer an arbitrary two-dimensional polynomial under any subgroup of the group of two-dimensional affine transforms. In [6], the authors show that such polynomials can be steered, with a finite number of basis functions, under any subgroup of the group of two-dimensional affine transforms. Thus, the procedure is guaranteed to terminate after a finite number of iterations.

In the procedure, the linear independence of a polynomial with respect to the current basis set needs to be determine. This is done by representing each polynomial in terms of the basis of monomials $\{1, x, y, x^2, xy, y^2, \ldots\}$. Specifically, let the matrix $\boldsymbol{B}$ be an $m \times n$ matrix of coefficients and $\boldsymbol{m}$ be the $n \times 1$ vector of monomials such that $\boldsymbol{Bm}$ yields an $m \times 1$ vector corresponding to the m basis polynomials. Similarly, expressing the new polynomial in the monomial basis results in a $1 \times n$ vector $\boldsymbol{b}$ of coefficients. Since the monomials are linearly independent, the new polynomial is linearly dependent on the basis set if and only if $\boldsymbol{b}$ is in the row space of $\boldsymbol{B}$. The generators for each one-parameter subgroup (e.g. x-translation, y-translation, etc.)

are implemented as operations on the coefficients of the polynomial representation. This is possible since applying any generator to a polynomial always results in another polynomial.

The same cubic polynomial $x^3+3x^2y+3xy^2+y^3$ (`mpoly2`) is used in all the examples. The basis functions needed to steer the function under different multi-parameter groups are computed.

1. In this example, basis functions to steer the polynomial under the group of uniform scaling and rotation are computed. The generators are $L_s = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ and $L_r = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ respectively.

   `poly2_mat = steer_poly2(mypoly2, 'lscale', 'lrot')`

$$
\begin{aligned}
&x^3 + 3x^2y + 3xy^2 + y^3, \\
&3x^3 + 3x^2y - 3xy^2 - 3y^3, \\
&3x^3 - 15x^2y - 15xy^2 + 3y^3, \\
&-15x^3 - 39x^2y + 39xy^2 + 15y^3.
\end{aligned}
$$

2. In this example, basis functions to steer the polynomial under the group of translations in the x and y directions are computed. The generators are $L_x = -\frac{\partial}{\partial x}$ and $L_y = -\frac{\partial}{\partial y}$ respectively.

   `poly2_mat = steer_poly2(mypoly2, 'ltransx', 'ltransy')`

$$
\begin{aligned}
&x^3 + 3x^2y + 3xy^2 + y^3, \\
&-3x^2 - 6xy - 3y^2, \\
&6x + 6y, \\
&-6.
\end{aligned}
$$

3. In this example, basis functions to steer the polynomial under the group of translations in the x and y directions and rotation are computed. The generators are $L_x = -\frac{\partial}{\partial x}$, $L_y = -\frac{\partial}{\partial y}$, and $L_r = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ respectively.

   `poly2_mat = steer_poly2(mypoly2, 'ltransx', 'ltransy', 'lrot')`

$$
\begin{aligned}
&-15x^3 - 39x^2y + 39xy^2 + 15y^3, \\
&3x^3 - 15x^2y - 15xy^2 + 3y^3, \\
&3x^3 + 3x^2y - 3xy^2 - 3y^3, \\
&x^3 + 3x^2y + 3xy^2 + y^3, \\
&-3x^2 - 6xy - 3y^2,
\end{aligned}
$$

$$-6x^2 + 6y^2,$$
$$6x + 6y,$$
$$6x - 6y,$$
$$24xy,$$
$$-6.$$

Clearly, the basis comprising of all the monomials in x, y up to powers of three, a total of ten, is sufficient to steer the cubic polynomial. However, as can be seen in the examples above, fewer than ten are actually needed in some situations. The procedure selectively retains only those necessary by removing those that are linearly dependent (with respect to the generators of the group).

5.2 Numerical Simulations

A numerical version of the procedure in Figure 2 was also implemented. The program automatically computes a set of basis functions that can be used to steer a given two-dimensional function. The derivatives in the generators were approximated using numerical derivatives. The linear dependence of a function on the current set of basis functions is verified by projecting the function onto an orthogonalized version of the basis set and measuring the relative magnitude of the residual. The set of orthogonal basis functions can be efficiently computed by using the Gram-Schmidt technique iteratively.

Since the procedure is not guaranteed to terminate for arbitrary functions as an infinite number of basis functions might be required, the check for linear dependence was replaced by a numerical condition that the residual between the function and its projection is below some threshold. Furthermore, the maximum depth of the generator tree was also used as a termination criteria since higher-order numerical derivatives are less accurate. As a result, the steering of the given function with the basis set is only accurate to within some range of transform parameters as we shall see.

Figure 3 shows images of four basis functions that could be used to steer (under rotation) the function $(12x - 8x^3)\exp[-(x^2 + y^2)]$, which is the third derivative of a Gaussian. The leftmost image is the function that was used as input to the procedure; i.e., the function to be steered. The spatial extent of each image ranges from -5 to 5 units both horizontally and vertically. Figure 4 shows images of four orthogonal basis functions that could also be used to steer the function. These basis functions were computed by the Gram-Schmidt component of the procedure.

Unlike the third derivative of a Gaussian, the function $\sin(x)\exp[-(x^2 + y^2)]$ cannot be perfectly steered under rotation. Figure 5 shows images of the four basis functions returned by the procedure. The maximum tree-depth was set at 6 and the maximum relative squared error of the residual was 5%. The relative squared error of using these basis functions to steer the function under any rotation was always less than 0.1%. Figure 6 shows images of four orthogonal basis functions that could also be used to steer the function.

Figure 7 shows images of 22 basis functions that could be used to steer the function $(4x^2-2)\exp[-(x^2+y^2)]$ under any x, y translation and rotation. Figure 8 shows images of the orthogonalized basis functions. Again, the steering is only approximate since the function cannot be steered with a finite number of basis functions. The maximum tree-depth of the procedure was set at 3 and the maximum relative squared error of the residual was 10%. Figure 9 plots the relative squared errors of steering the function using this basis for a range of translations. The approximation is very good about the origin (zero translation) and worsens when the translations are large. Figure 10 plots the relative squared errors of steering a translated version of the function over all rotation angles. The errors in steering the untranslated function are negligible since the second derivative of a Gaussian can be perfectly steered with three basis functions under orientation.

6 Discussion

The proposed method for computing basis functions to steer a given function essentially computes the Taylor expansion of the function with respect to the transform parameters. The expansion is evaluated at the origin of the transform parameters. Several simplifications arise because the transform is a Lie transformation group. The principal one being that the Taylor expansion can be written solely in terms of the first order derivatives, namely, the generators. Equation 4 of Section 3 and Equation 5 of Section 4 describe the Taylor expansion in terms of the generator(s) for one-parameter and multi-parameter groups respectively. Since higher order derivatives can be determined from these generators, properties involving the higher order derivatives can be proven. These properties are precisely those that were used to show the minimality of generator chains and generator trees. Furthermore, these higher order derivatives can also be computed by repeated applications of the generators.

As a result of this close connection with Taylor expansions, the errors incurred in approximately steering a function increases with the deviation of the transform parameters from the origin. This happens when the function to be steered cannot be steered by a finite number of basis functions. This can be seen from the numerical implementation of the procedure in Section 5. This may be acceptable for some applications where only a limited range of steering is required. For example, Manmatha [8] uses a similar

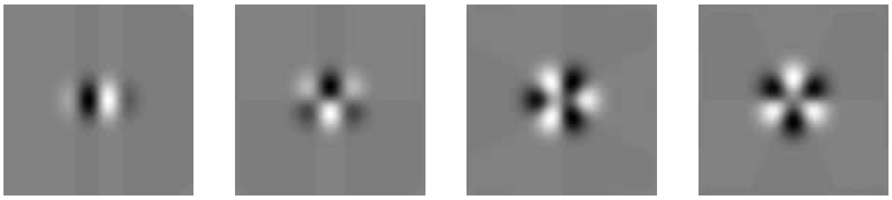


Figure 3: Basis functions that steer $(12x - 8x^3)\exp[-(x^2 + y^2)]$, the third derivative of a Gaussian, under rotation. The leftmost image is the third derivative of a Gaussian that was used as input to the procedure.

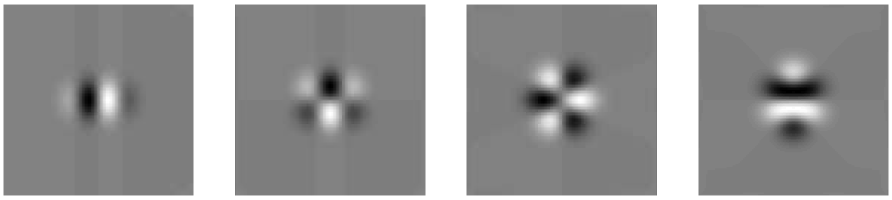


Figure 4: Orthogonal basis functions that steer $(12x - 8x^3)\exp[-(x^2 + y^2)]$, the third derivative of a Gaussian, under rotation. The leftmost image is the third derivative of a Gaussian that was used as input to the procedure.

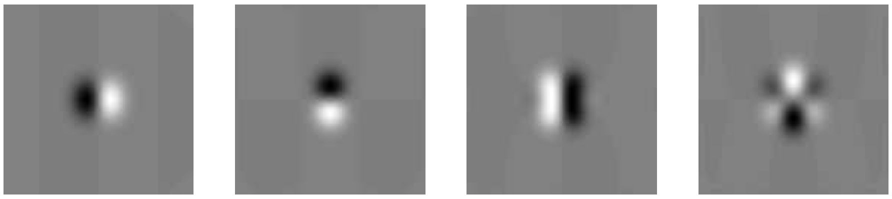


Figure 5: Basis functions that steer $\sin(x)\exp[-(x^2 + y^2)]$ under rotation. The leftmost image is the function that was used as input to the procedure.

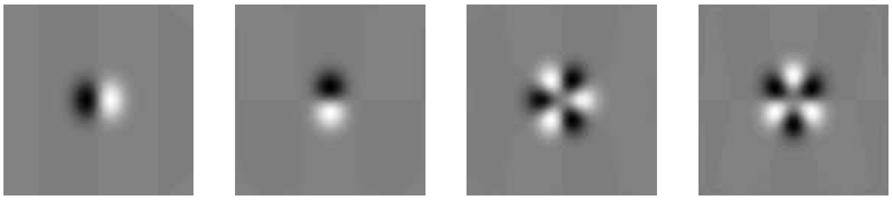


Figure 6: Orthogonal basis functions that steer $\sin(x)\exp[-(x^2 + y^2)]$ under rotation. The leftmost image is the function that was used as input to the procedure.

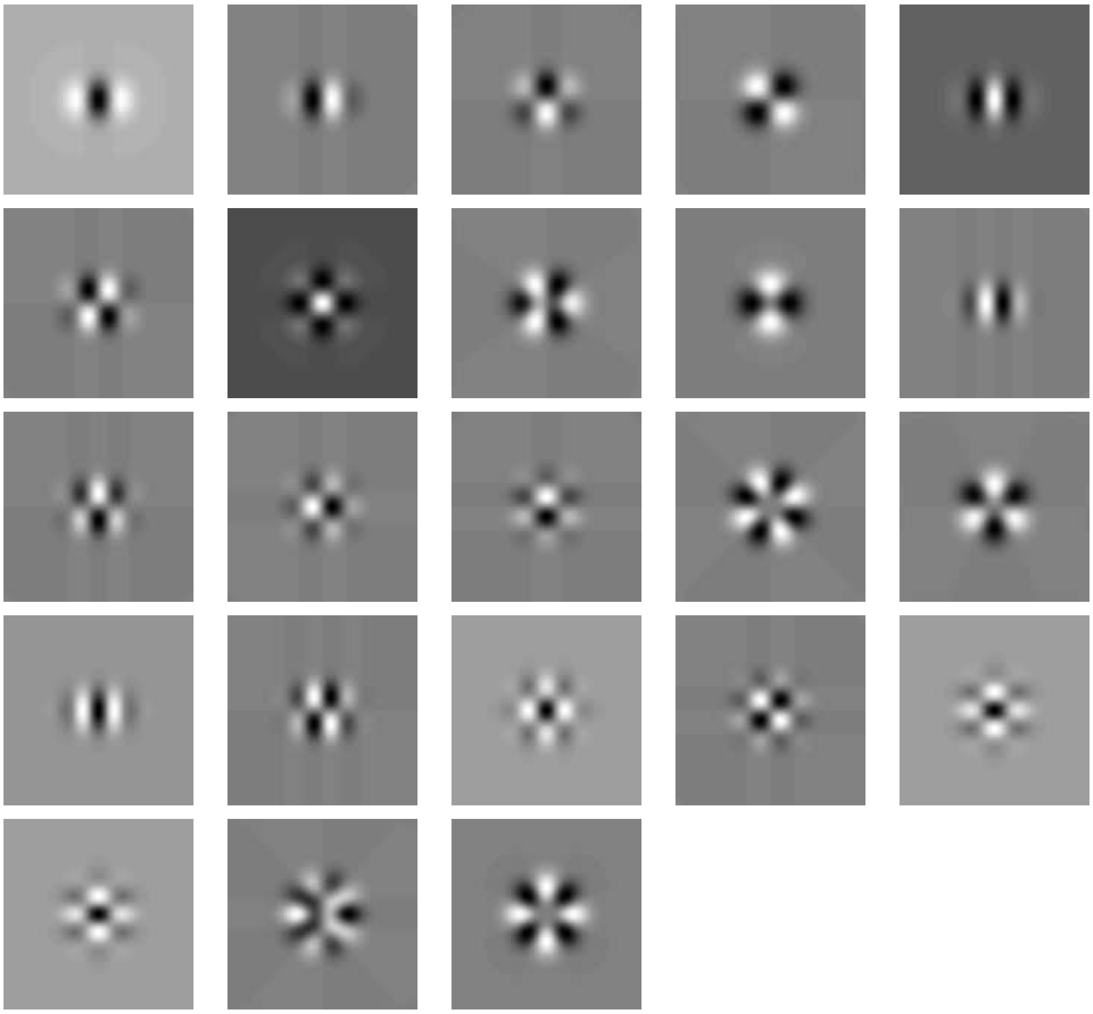


Figure 7: Basis functions that steer $(4x^2 - 2)\exp[-(x^2 + y^2)]$ under $x, y-$ translation and rotation. The image at the top left is the function that was used as input to the procedure.

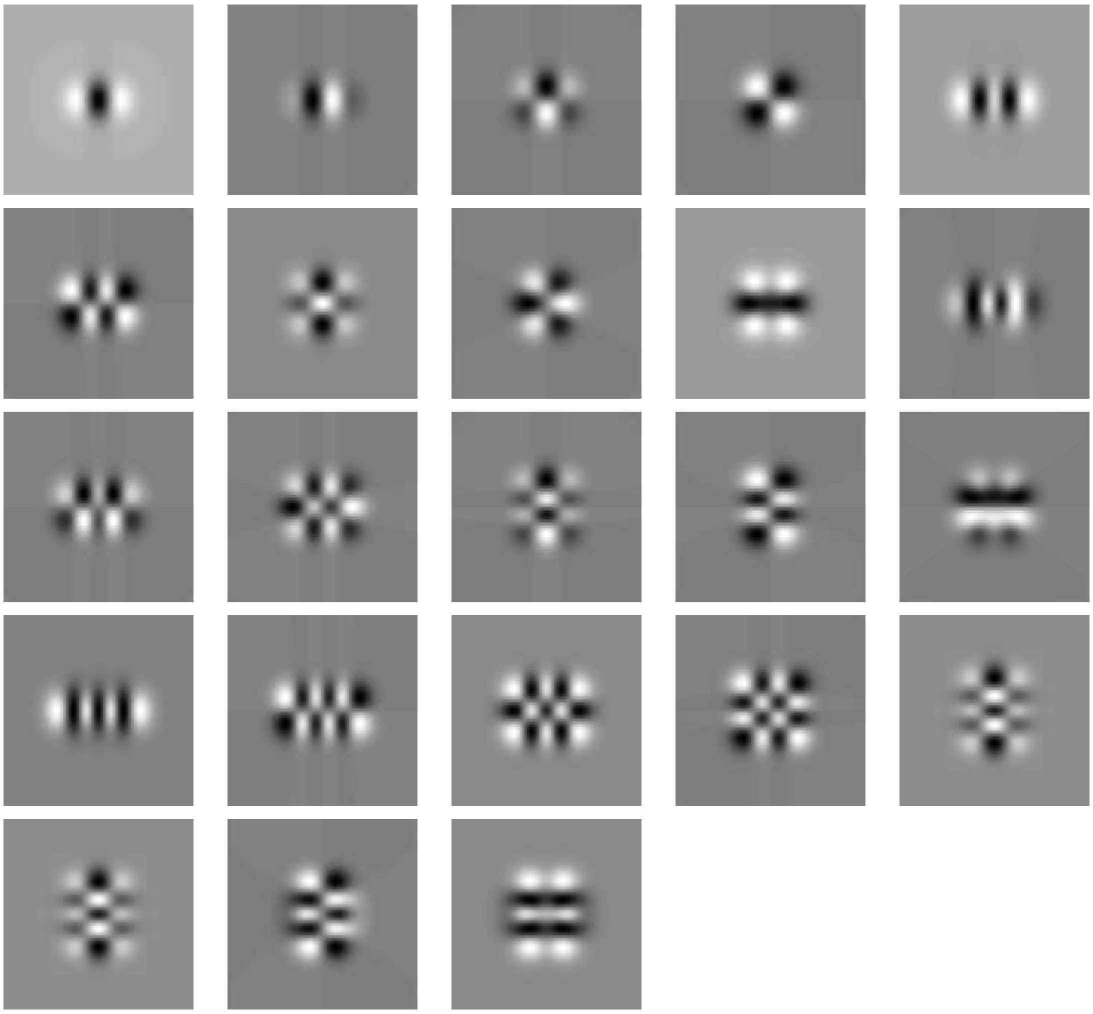
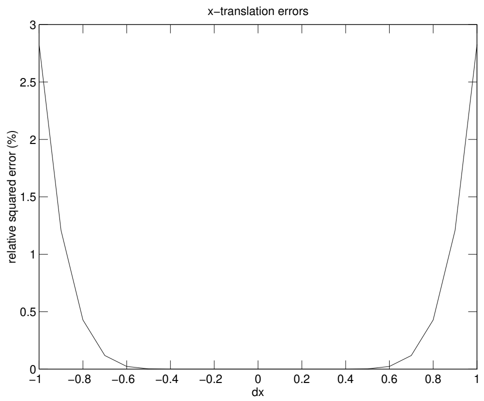
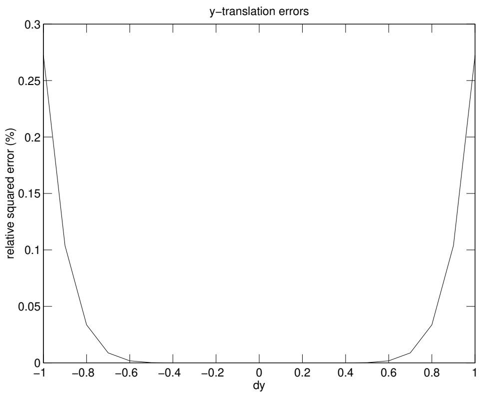


Figure 8: Orthogonal basis functions that steer $(4x^2-2)\exp[-(x^2+y^2)]$ under $x, y-$ translation and rotation. The image at the top left is the function that was used as input to the procedure.



(a)



(b)

Figure 9: Relative squared errors of the steered approximations to the actual functions over a range of translations. Graph (a) plots the errors for translations along the x-dimension. Graph (b) plots the errors for translations along the y-dimension.

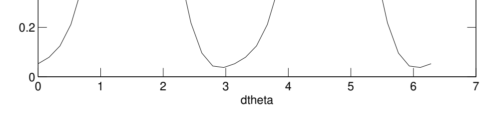


Figure 10: Relative squared errors of the steered approximations to the actual functions over the entire range of rotation angles. The actual function has been translated by 0.5 units in both the x and y dimensions; i.e., $(4(x-0.5)^2-2)\exp[-((x-0.5)^2+(y-0.5)^2)]$. The percentage errors for an untranslated function are negligible.

approach to estimate the affine transformation of points, lines and image intensities. However, if the function needs to be steered over a larger range of parameters, then either more basis functions could be computed by increasing the maximum tree-depth or by applying the Taylor expansion about another location other than the origin. The basis functions computed by this method, in fact, minimizes the approximation error about the particular transform parameter. Instead, if the criterion is to minimize the average error over a range of transform parameters, then fewer basis functions are required. In [14], the authors propose a method of computing the basis functions that minimizes this approximation error.

If the function to be steered is obtained from a space of functions that is steerable, then the function can be steered with a finite number of basis functions. Consequently, an analytic version of the procedure could be applied to determine the smallest basis set required. The example for polynomials is illustrated in Section 5. As shown in that section, while each monomial could be used as a basis function to steer the given polynomial, often fewer basis functions are sufficient because of various linear dependencies. A similar algorithm could also be implemented for sinusoids over translation/rotation or spherical harmonics over $3D$ rotation.

In summary, we have presented a computational, group-theoretic approach to computing the basis functions of steerable functions. If the function is steerable with n basis functions under a k-parameter group, the procedure is efficient in that at most $nk+1$ iterations of the procedure are needed to compute all the basis functions. Furthermore, the procedure is guaranteed to return the minimum number of basis functions. If the function is not steerable, a numerical implementation of the procedure could be used to compute basis functions that approximately steer the function over a range of parameters.

References

[1] A. Cohen. *An introduction to the Lie theory of one-parameter groups; with applications to the solution of differential equations.* D. C. Heath & Co., Boston; New York, 1911.

[2] D. Fleet. Computation of component image velocity from local phase information. *International Journal of Computer Vision*, 5(1):77–104, 1990.

[3] W. Freeman and E. Adelson. The design and use of steerable filters. *IEEE Trans. Pattern Analysis and Machine Intelligence*, 13(9):891–906, 1991.

[4] C. Gotsman. Constant-time filtering by singular value decomposition. *Computer Graphics Forum*, 13(2):153–163, 1994.

[5] G. Granlund and H. Knutsson. *Signal processing for computer vision.* Kluwer Academic Publishers, Boston, 1995.

[6] Y. Hel-Or and P. Teo. A common framework for steerability, motion estimation and invariant feature detection. Technical Report STAN-CS-TN-96-28, Stanford University, 1996.

[7] H. Liu, T. Hong, M. Herman, and R. Chellappa. A reliable optical flow algorithm using 3-d hermite polynomials. Technical Report CS-TR-3291, University of Maryland, 1994.

[8] R. Manmatha. A framework for recovering affine transforms using points, lines or image brightnesses. In *Proc. IEEE Conf. Computer Vision and Pattern Recognition*, pages 141–146, Seattle, WA, 1994.

[9] M. Michaelis and G. Sommer. A Lie group-approach to steerable filters. *Pattern Recognition Letters*, 16(11):1165–1174, November 1995.

[10] J Nimeroff, E Simoncelli, and J Dorsey. Efficient re-rendering of naturally illuminated environments. In *5th Eurographics Workshop on Rendering*, 1994.

[11] P. Perona. Deformable kernels for early vision. *IEEE Trans. Pattern Analysis and Machine Intelligence*, 17(5):488–499, 1995.

[12] E. Simoncelli and H. Farid. Steerable wedge filters. In *Proc. Int. Conf. on Computer Vision*, pages 189–194, Boston, MA, 1995.

[13] E. Simoncelli, W. Freeman, E. Adelson, and D. Heeger. Shiftable multiscale transforms. *IEEE Trans. Information Theory*, 38(2):587–607, 1992.

[14] P. Teo and Y. Hel-Or. Design of multi-parameter steerable functions using cascade basis reduction. Technical Report STAN-CS-TN-96-32, Stanford University, 1996.

[15] J. Weng. Image matching using the windowed fourier phase. *Int. J. Computer Vision*, 11(3):211–236, 1993.

[16] Y. Xiong and S. Shafer. Moment and hypergeometric filters for high precision computation of focus, stereo and optical flow. Technical Report CMU-RI-TR-94-28, Carnegie Mellon University, 1994.

Appendix:

A Number of Iterations

Claim 1 : The procedure given in Figure 2 of Section 4 tests the linear dependence of a function on the basis set at most $nk+1$ times for a function that is steerable with n basis functions under a k-parameter group.

Before proving this claim, we proof the following useful lemma.

Lemma 1 : A k-ary tree with n nodes (internal nodes as well as leaves) has exactly $n(k-1)+1$ immediate children.

Proof of Lemma 1: A k-ary tree with one node has k immediate children. Each addition of a new node increases the number of immediate children by $k-1$. Thus, adding $n-1$ nodes results in a total number of $(n-1)(k-1)+k = nk+1$ immediate children in the tree.

Proof of Claim 1 : The number of times the linear dependence test is invoked is equal to the sum of the number of basis functions required and the number of immediate children in the resultant k-ary generator tree. Therefore, the total number of times the test is applied is $n + n(k-1) + 1 = nk + 1$. $\square$