Closure Properties of Regular Languages

Union, Intersection, Difference, Concatenation, Kleene Closure, Reversal, Homomorphism, Inverse Homomorphism

Closure Properties

- Recall a closure property is a statement that a certain operation on languages, when applied to languages in a class (e.g., the regular languages), produces a result that is also in that class.
- For regular languages, we can use any of its representations to prove a closure property.

Closure Under Union

- If L and M are regular languages, so is $L \cup M$.
- Proof: Let L and M be the languages of regular expressions R and S, respectively.
- Then $R+S$ is a regular expression whose language is $L \cup M$.

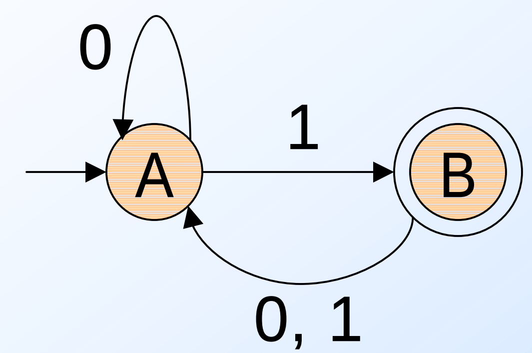
Closure Under Concatenation and Kleene Closure

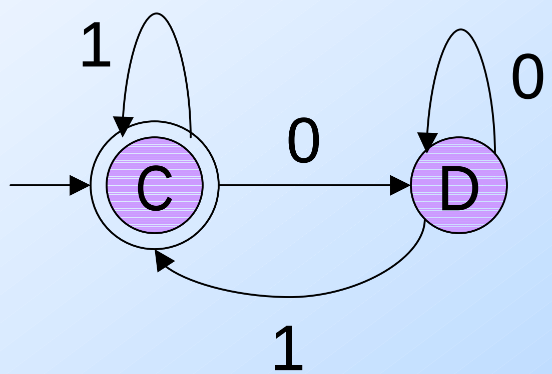
- Same idea:
  - RS is a regular expression whose language is LM.
  - R* is a regular expression whose language is L*.

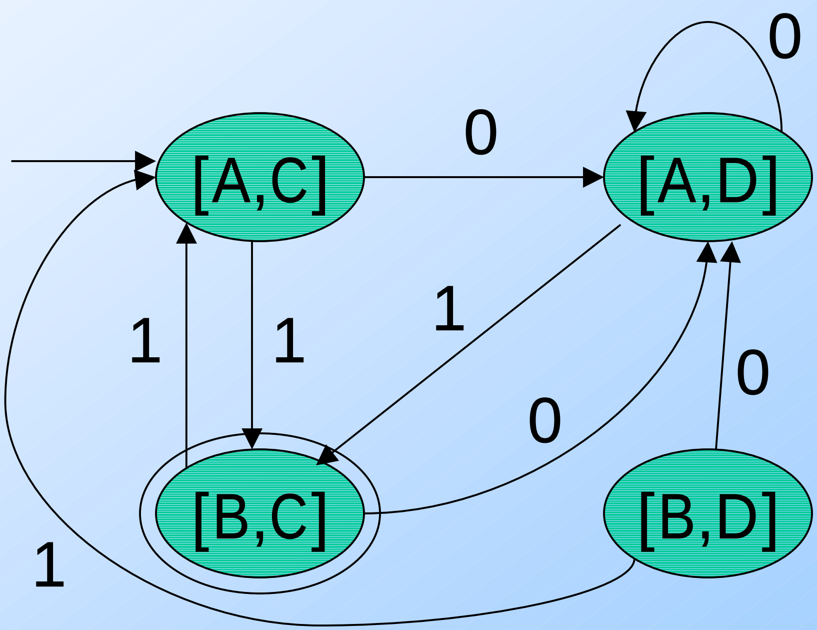
Closure Under Intersection

- If L and M are regular languages, then so is $L \cap M$.
- Proof: Let A and B be DFA's whose languages are L and M, respectively.
- Construct C, the product automaton of A and B.
- Make the final states of C be the pairs consisting of final states of both A and B.

Example: Product DFA for Intersection



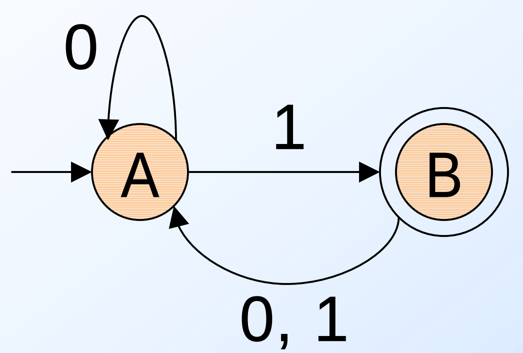


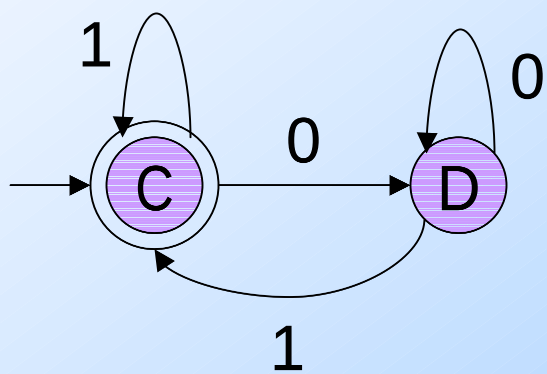


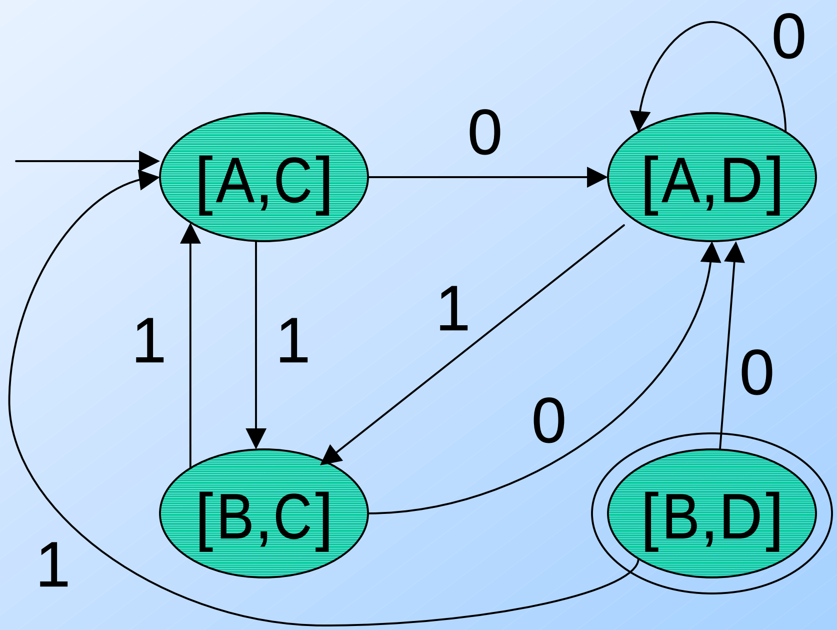
Closure Under Difference

- If L and M are regular languages, then so is $L - M$ = strings in L but not M.
- Proof: Let A and B be DFA's whose languages are L and M, respectively.
- Construct C, the product automaton of A and B.
- Make the final states of C be the pairs where A-state is final but B-state is not.

Example: Product DFA for Difference







Notice: difference is the empty language

Closure Under Complementation

- The *complement* of a language L (with respect to an alphabet Σ such that Σ^* contains L) is $\Sigma^* - L$.
- Since Σ^* is surely regular, the complement of a regular language is always regular.

Closure Under Reversal

- Recall example of a DFA that accepted the binary strings that, as integers were divisible by 23.
- We said that the language of binary strings whose reversal was divisible by 23 was also regular, but the DFA construction was very tricky.
- Good application of reversal-closure.

Closure Under Reversal – (2)

- Given language L, L^R is the set of strings whose reversal is in L.
- Example: L = {0, 01, 100};
L^R = {0, 10, 001}.
- Proof: Let E be a regular expression for L.
- We show how to reverse E, to provide a regular expression E^R for L^R.

Reversal of a Regular Expression

- Basis: If E is a symbol a, ϵ, or ∅, then $E^R = E$.
- Induction: If E is
  - F+G, then $E^R = F^R + G^R$.
  - FG, then $E^R = G^R F^R$
  - F*, then $E^R = (F^R)^*$.

Example: Reversal of a RE

- Let E = $\mathbf{01^* + 10^*}$.
- $E^R = (\mathbf{01^* + 10^*})^R = (\mathbf{01^*})^R + (\mathbf{10^*})^R$
- $= (\mathbf{1^*})^R\mathbf{0}^R + (\mathbf{0^*})^R\mathbf{1}^R$
- $= (\mathbf{1}^R)^*\mathbf{0} + (\mathbf{0}^R)^*\mathbf{1}$
- $= \mathbf{1^*0 + 0^*1}$.

Homomorphisms

- A *homomorphism* on an alphabet is a function that gives a string for each symbol in that alphabet.
- Example: $h(0) = ab$; $h(1) = \epsilon$.
- Extend to strings by $h(a_1\ldots a_n) = h(a_1)\ldots h(a_n)$.
- Example: $h(01010) = ababab$.

Closure Under Homomorphism

- If L is a regular language, and h is a homomorphism on its alphabet, then h(L) = {h(w) | w is in L} is also a regular language.
- Proof: Let E be a regular expression for L.
- Apply h to each symbol in E.
- Language of resulting RE is h(L).

Example: Closure under Homomorphism

- Let $h(0) = ab$; $h(1) = \epsilon$.
- Let L be the language of regular expression **01*** + **10***.
- Then h(L) is the language of regular expression **ab**ϵ* + ϵ(**ab**)*.

Note: use parentheses to enforce the proper grouping.

Example – Continued

- **ab**ϵ* + ϵ(**ab**)* can be simplified.
- ϵ* = ϵ, so **ab**ϵ* = **ab**ϵ.
- ϵ is the identity under concatenation.
  - That is, ϵE = Eϵ = E for any RE *E*.
- Thus, **ab**ϵ* + ϵ(**ab**)* = **ab**ϵ + ϵ(**ab**)* = **ab** + (**ab**)*.
- Finally, L(**ab**) is contained in L((**ab**)*), so a RE for h(L) is (**ab**)*.

Inverse Homomorphisms

- Let h be a homomorphism and L a language whose alphabet is the output language of h.
- $h^{-1}(L)$ = {w | h(w) is in L}.

Example: Inverse Homomorphism

- Let $h(0) = ab$; $h(1) = \epsilon$.
- Let $L = \{abab, baba\}$.
- $h^{-1}(L)$ = the language with two 0's and any number of 1's = $L(\mathbf{1^*01^*01^*})$.

Notice: no string maps to baba; any string with exactly two 0's maps to abab.

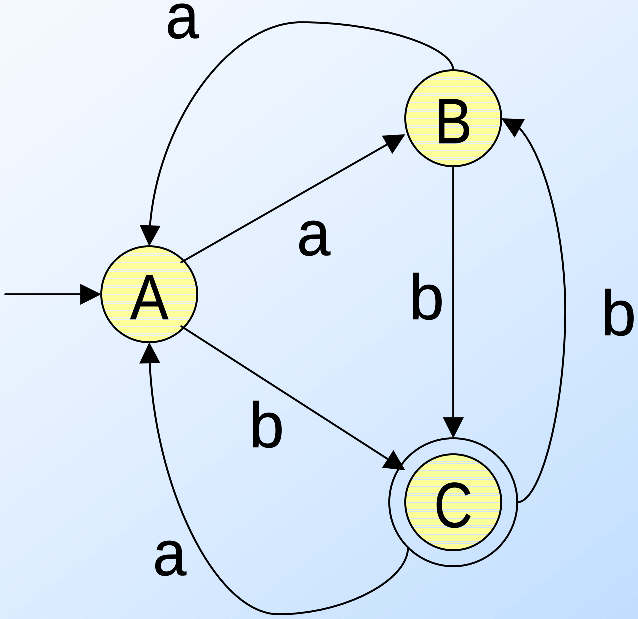
Closure Proof for Inverse Homomorphism

- Start with a DFA A for L.
- Construct a DFA B for $h^{-1}(L)$ with:
  - The same set of states.
  - The same start state.
  - The same final states.
  - Input alphabet = the symbols to which homomorphism h applies.

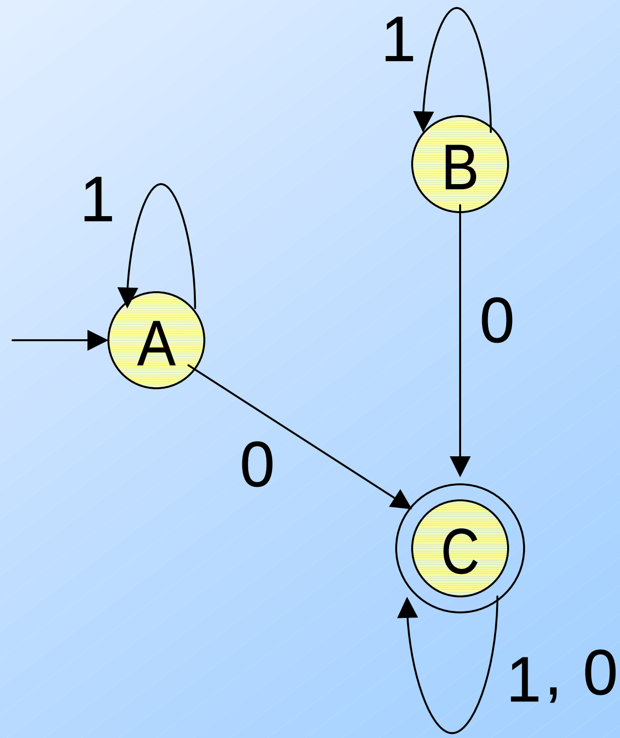
Proof – (2)

- The transitions for B are computed by applying h to an input symbol *a* and seeing where A would go on sequence of input symbols h(a).
- Formally, $\delta_B(q, a) = \delta_A(q, h(a))$.

Example: Inverse Homomorphism Construction



$h(0) = ab$
$h(1) = \epsilon$



Since
$h(1) = \epsilon$

Since
$h(0) = ab$

Proof – (3)

- Induction on $|w|$ shows that $\delta_B(q_0, w) = \delta_A(q_0, h(w))$.
- Basis: $w = \epsilon$.
- $\delta_B(q_0, \epsilon) = q_0$, and $\delta_A(q_0, h(\epsilon)) = \delta_A(q_0, \epsilon) = q_0$.

Proof – (4)

- Induction: Let $w = xa$; assume IH for x.
- $\delta_B(q_0, w) = \delta_B(\delta_B(q_0, x), a)$.
- $= \delta_B(\delta_A(q_0, h(x)), a)$ by the IH.
- $= \delta_A(\delta_A(q_0, h(x)), h(a))$ by definition of the DFA B.
- $= \delta_A(q_0, h(x)h(a))$ by definition of the extended delta.
- $= \delta_A(q_0, h(w))$ by def. of homomorphism.