Intractable Problems

Time-Bounded Turing Machines

Classes **P** and **NP**

Polynomial-Time Reductions

Time-Bounded TM's

- A Turing machine that, given an input of length n, always halts within T(n) moves is said to be *T(n)-time bounded*.
  - The TM can be multitape.
  - Sometimes, it can be nondeterministic.
- The deterministic, multitape case corresponds roughly to "an O(T(n)) running-time algorithm."

The class **P**

- If a DTM M is T(n)-time bounded for some polynomial T(n), then we say M is *polynomial-time* ("*polytime*") bounded.
- And L(M) is said to be in the class **P**.
- Important point: when we talk of **P**, it doesn't matter whether we mean "by a computer" or "by a TM" (next slide).

Polynomial Equivalence of Computers and TM's

- A multitape TM can simulate a computer that runs for time $O(T(n))$ in at most $O(T^2(n))$ of its own steps.
- If $T(n)$ is a polynomial, so is $T^2(n)$.

Examples of Problems in **P**

- Is w in L(G), for a given CFG G?
  - Input = w.
  - Use CYK algorithm, which is $O(n^3)$.
- Is there a path from node x to node y in graph G?
  - Input = x, y, and G.
  - Use Dijkstra's algorithm, which is O(n log n) on a graph of n nodes and arcs.

Running Times Between Polynomials

- You might worry that something like O(n log n) is not a polynomial.
- However, to be in **P**, a problem only needs an algorithm that runs in time less than some polynomial.
- Surely O(n log n) is less than the polynomial $O(n^2)$.

A Tricky Case: Knapsack

- The *Knapsack Problem* is: given positive integers i_1, i_2 ,..., i_n, can we divide them into two sets with equal sums?
- Perhaps we can solve this problem in polytime by a dynamic-programming algorithm:
  - Maintain a table of all the differences we can achieve by partitioning the first j integers.

Knapsack – (2)

- Basis: $j = 0$. Initially, the table has "true" for 0 and "false" for all other differences.
- Induction: To consider i_j, start with a new table, initially all false.
- Then, set k to true if, in the old table, there is a value m that was true, and k is either $m+i_j$ or $m-i_j$.

Knapsack – (3)

- Suppose we measure running time in terms of the sum of the integers, say m.
- Each table needs only space O(m) to represent all the positive and negative differences we could achieve.
- Each table can be constructed in time O(n).

Knapsack – (4)

- Since $n \leq m$, we can build the final table in $O(m^2)$ time.
- From that table, we can see if 0 is achievable and solve the problem.

Subtlety: Measuring Input Size

- "Input size" has a specific meaning: the length of the representation of the problem instance as it is input to a TM.
- For the Knapsack Problem, you cannot always write the input in a number of characters that is polynomial in either the number-of or sum-of the integers.

Knapsack – Bad Case

- Suppose we have n integers, each of which is around 2^n.
- We can write integers in binary, so the input takes $O(n^2)$ space to write down.
- But the tables require space $O(n2^n)$.
- They therefore require at least that order of time to construct.

Bad Case – (2)

- Thus, the proposed "polynomial" algorithm actually takes time $O(n^2 2^n)$ on an input of length $O(n^2)$.
- Or, since we like to use n as the input size, it takes time $O(n2^{sqrt(n)})$ on an input of length n.
- In fact, it appears no algorithm solves Knapsack in polynomial time.

Redefining Knapsack

- We are free to describe another problem, call it *Pseudo-Knapsack*, where integers are represented in unary.
- Pseudo-Knapsack is in **P**.

The Class **NP**

- The running time of a nondeterministic TM is the maximum number of steps taken along any branch.
- If that time bound is polynomial, the NTM is said to be *polynomial-time bounded*.
- And its language/problem is said to be in the class **NP**.

Example: **NP**

- The Knapsack Problem is definitely in **NP**, even using the conventional binary representation of integers.
- Use nondeterminism to guess one of the subsets.
- Sum the two subsets and compare.

P Versus **NP**

- Originally a curiosity of Computer Science, mathematicians now recognize as one of the most important open problems the question **P** = **NP**?
- There are thousands of problems that are in **NP** but appear not to be in **P**.
- But no proof that they aren't really in **P**.

Complete Problems

- One way to address the **P** = **NP** question is to identify *complete problems* for NP.
- An *NP-complete problem* has the property that if it is in **P**, then every problem in **NP** is also in **P**.
- Defined formally via "polytime reductions."

Complete Problems – Intuition

- A complete problem for a class embodies every problem in the class, even if it does not appear so.
- Compare: PCP embodies every TM computation, even though it does not appear to do so.
- Strange but true: Knapsack embodies every polytime NTM computation.

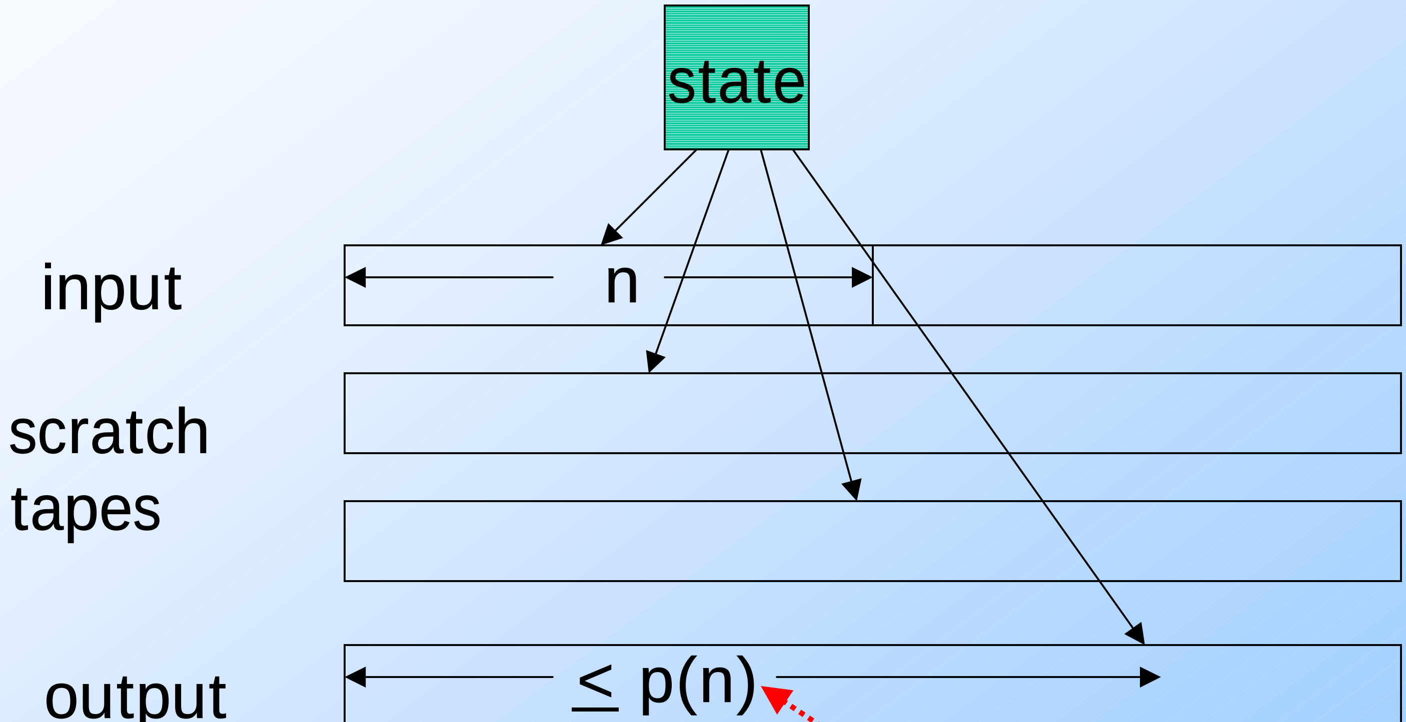
Polytime Reductions

- Goal: find a way to show problem L to be NP-complete by reducing every language/problem in **NP** to L in such a way that if we had a deterministic polytime algorithm for L, then we could construct a deterministic polytime algorithm for any problem in **NP**.

Polytime Reductions – (2)

- We need the notion of a *polytime transducer* – a TM that:
  1. Takes an input of length n.
  2. Operates deterministically for some polynomial time p(n).
  3. Produces an output on a separate *output tape*.
- Note: output length is at most p(n).

Polytime Transducer

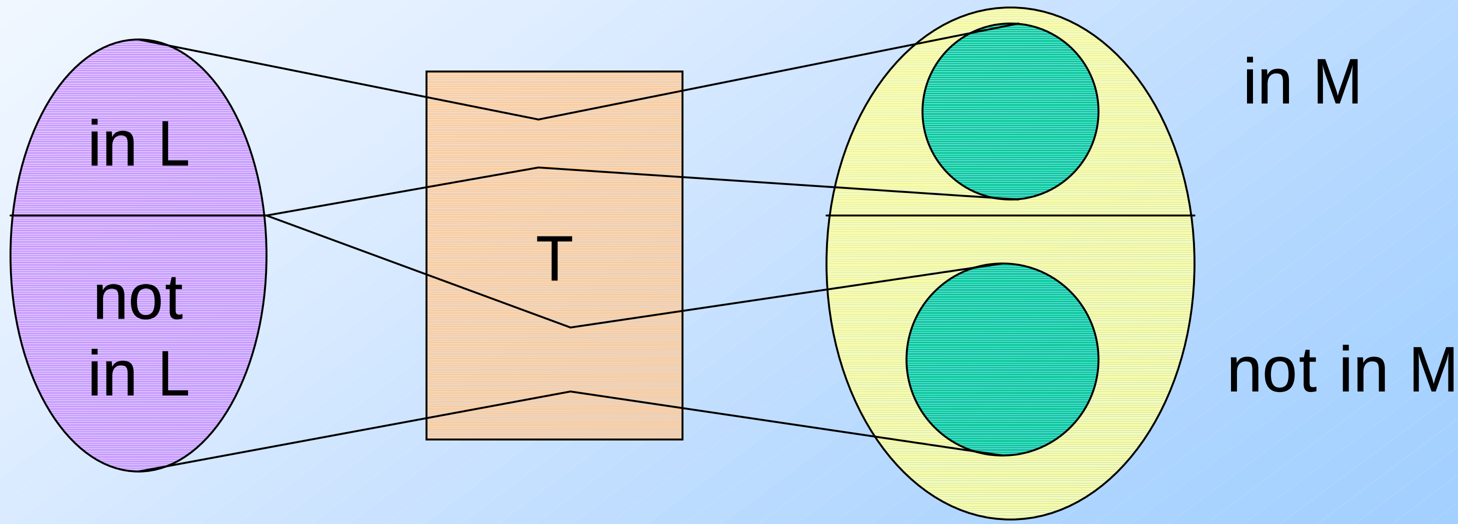


Remember: important requirement is that *time* ≤ p(n).

Polytime Reductions – (3)

- Let L and M be langauges.
- Say L is *polytime reducible* to M if there is a polytime transducer T such that for every input w to T, the output x = T(w) is in M if and only if w is in L.

Picture of Polytime Reduction



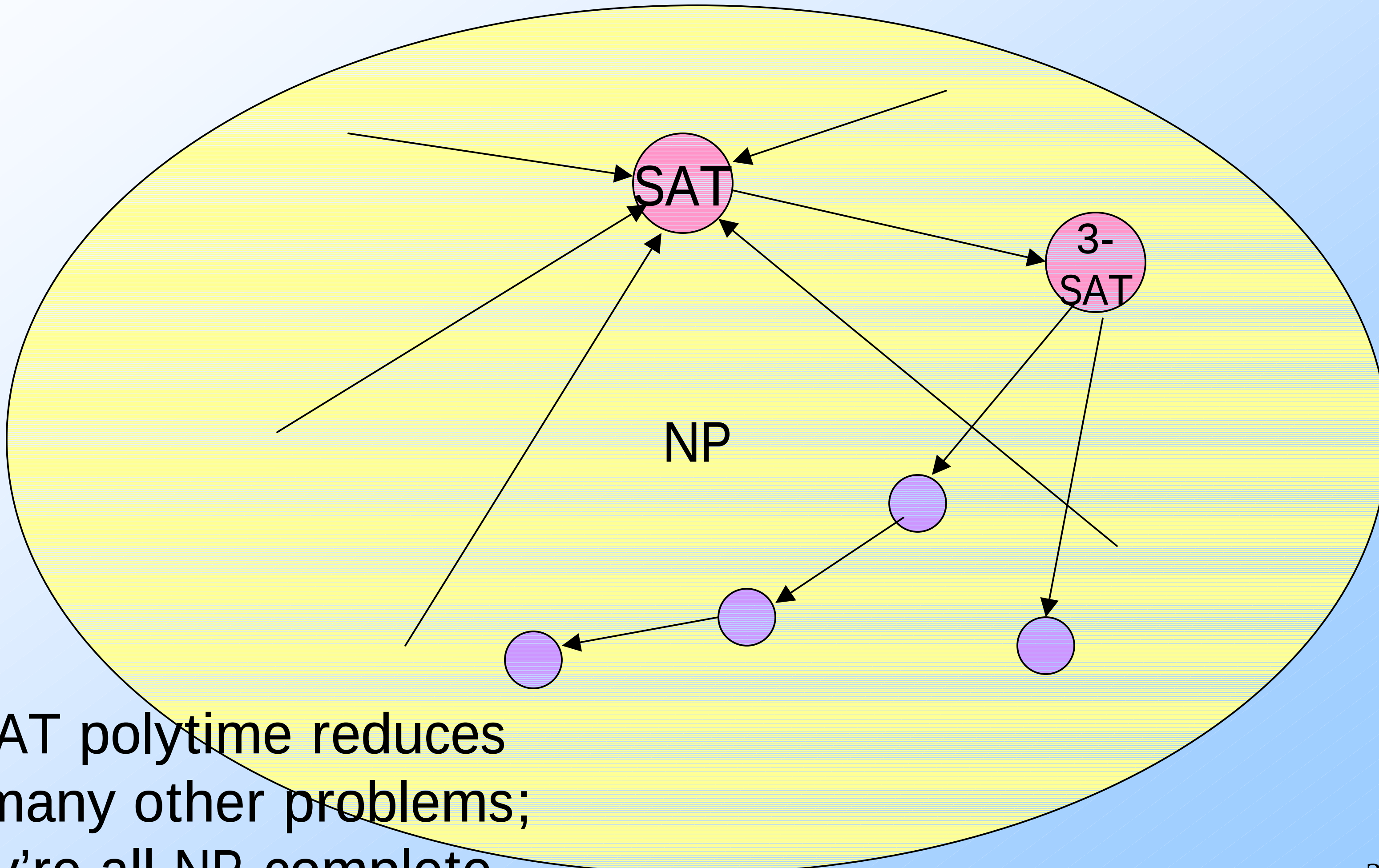
NP-Complete Problems

- A problem/language M is said to be *NP-complete* if for every language L in **NP**, there is a polytime reduction from L to M.
- Fundamental property: if M has a polytime algorithm, then L also has a polytime algorithm.
  - I.e., if M is in **P**, then every L in **NP** is also in **P**, or "**P** = **NP**."

The Plan

All of **NP** polytime reduces to SAT, which is therefore NP-complete

SAT polytime reduces to 3-SAT



3-SAT polytime reduces to many other problems; they're all NP-complete

Proof That Polytime Reductions "Work"

- Suppose M has an algorithm of polynomial time q(n).
- Let L have a polytime transducer T to M, taking polynomial time p(n).
- The output of T, given an input of length n, is at most of length p(n).
- The algorithm for M on the output of T takes time at most q(p(n)).

Proof – (2)

- We now have a polytime algorithm for L:
  1. Given w of length n, use T to produce x of length $\leq$ p(n), taking time $\leq$ p(n).
  2. Use the algorithm for M to tell if x is in M in time $\leq$ q(p(n)).
  3. Answer for w is whatever the answer for x is.
- Total time $\leq$ p(n) + q(p(n)) = a polynomial.