**CS109A Notes for Lecture 1/17/96**

## Simple Inductions

Three pieces:

1. A statement $S(n)$ to be proved.
   - □ The statement must be about an integer parameter $n$.
2. A *basis* for the proof. This is the statement $S(b)$ for some integer $b$. Often $b = 0$ or $b = 1$.
3. An *inductive step* for the proof. We prove the statement "$S(n)$ implies $S(n+1)$" for any $n$.

- The statement $S(n)$, used in this proof, is called the *inductive hypothesis.*
- We conclude that $S(n)$ is true for all $n \geq b$.
  - □ $S(n)$ might *not* be true for some $n < b$.

**Example:** The limit of the sum

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \cdots$$

is 1. (Each term is 1 divided by the product of two consecutive integers.)

To prove this fact, we can prove the following statement about the finite prefixes of the sum:

$$S(n): \quad \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

**Basis:** The basis is the case $n = 1$, that is, we must prove $S(1)$, or

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1+1}$$

There is one term, for $i = 1$, so we find that the left and right sides of the $=$ sign evaluate to $1/2$.

- Thus, the basis is true.

**Induction:** To prove the induction, we must prove $S(n)$ implies $S(n+1)$. $S(n+1)$ is:

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n+1}{(n+1)+1}$$

- Key "trick": Express $S(n+1)$ using $S(n)$ and "something left over."
- In this case, the sum in $S(n+1)$ is the sum in $S(n)$ plus the "extra" term for $i = n+1$: $1/(n+1)(n+2)$.

That is, $S(n+1)$ can be written:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{(n+1)+1}$$

Use the inductive hypothesis. The sum is equal to $n/(n+1)$; that's what $S(n)$ says. Thus, we must prove:

$$\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{(n+1)+1}$$

That's simple algebra!

- Thus, the induction is proved, and we conclude $S(n)$ holds for all $n \geq$ the basis value, 1.

**General Pattern for (Simple) Inductive Proofs**

1. State what $S(n)$ is.
2. Explain intuitively what $n$ represents, e.g., "any positive integer" or "the length of a string."
3. Tell what value of $n$ is the basis value, say $n = b$.
4. Prove $S(b)$.
5. State that you are assuming $n \geq b$ and that $S(n)$ is true.

6. Prove $S(n+1)$ using these assumptions. You will surely have to use the "inductive hypothesis" $S(n)$ in the proof.

7. State that as a result of your proofs (4) and (6), you conclude $S(n)$ for all $n \geq b$.

### More General Inductive Proofs

- There can be more than one basis case.
- We can do a *complete* induction (or "strong" induction), in which or proof that $S(n+1)$ is true uses any of $S(b), S(b+1), \ldots, S(n)$, where $b$ is the lowest basis value.
- Page 51 of FCS has the general proof outline.

### An Example With Multiple Basis Cases, Complete Induction

We claim that every integer $\geq 24$ can be written as $5a + 7b$ for nonnegative integers $a$ and $b$.

- Note that some integers $< 24$ cannot be expressed this way, e.g., 16, 23.

Let $S(n)$ be the statement "$n = 5a + 7b$ for some $a \geq 0$ and $b \geq 0$."

**Basis:** The five basis cases are 24 through 28.

$$
\begin{aligned}
24 &= 5 \times 2 + 7 \times 2 \\
25 &= 5 \times 5 + 7 \times 0 \\
26 &= 5 \times 1 + 7 \times 3 \\
27 &= 5 \times 4 + 7 \times 1 \\
28 &= 5 \times 0 + 7 \times 4
\end{aligned}
$$

**Induction:** Let $n+1 \geq 29$. Then $n-4 \geq 24$, the lowest basis case.

- Thus, $S(n-4)$ is true, and we can write $n - 4 = 5a + 7b$ for some $a$ and $b$.
- Thus, $n+1 = 5(a+1) + 7b$, proving $S(n+1)$.
  - ☐ Do not be thrown by the fact that $a$ in the statement $S(n+1)$ is $a+1$ here. The statement calls for "any $a$."

□ Actually, this "complete" induction is not very complete; we only used one previous statement, $S(n-4)$. But $S(n)$ is not enough of an inductive hypothesis.

## Class Problem for Next Time

What is the sum of the first $n$ terms of the series $2+5+8+11+\cdots$?

- Figure out the answer, then prove you are right by induction on $n$.