CS109A Midterm

Friday, Feb. 14, 2:15 – 3:05 PM

Directions

The exam is *open book*; any written materials may be used. Answer all 6 questions on the exam paper itself. The total number of points is 100. You may make additional comments at the bottoms of pages or at the end if you feel the need to clarify or justify your answers. Do not forget to **sign the pledge** below.

I acknowledge and accept the honor code.

_____

Print your name here: _____

1	/20
2	/15
3	/15
4	/20
5	/15
6	/15
Total	/100

Problem 1: (20 points) Prove by induction on n that for all $n \geq 1$, the sum

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n+1)$$

(that is, $\sum_{i=1}^{n} i(i+1)$) equals $n(n+1)(n+2)/3$. To organize your proof, answer the following questions.

a) What is the statement $S(n)$ to be proved?

b) What is a suitable basis case? _____

c) Prove the basis case.

d) When we prove the inductive step, we assume $S(n)$. What must we prove, using this assumption?

e) Prove the inductive step.

Problem 2: (15 points) A certain class of binary trees $T_0, T_1, T_2, \ldots$ is defined recursively by the following:

BASIS: T_0 is an empty tree, and T_1 is a tree with a single node.

INDUCTION: For all $n \geq 2$, T_n consists of a root node whose left subtree is a copy of T_{n-1}, and whose right subtree is a copy of T_{n-2}.

a) In the space below, draw T_2, T_3, and T_4.

T_2 T_3 T_4

b) Let $N(n)$ be the number of nodes in binary tree T_n. Write a recurrence relation for $N(n)$. You need not solve the recurrence.

Problem 3: (15 points) Write an ML function `getith(i,L)` that takes an integer i and a list L, and returns the ith element on list L. To simplify the problem, you may assume that i is at least 1, and that i is no larger than the length of the list L, so that there *is* an ith element on the list.

Problem 4: (20 points) Consider the recurrence

BASIS: $T(1) = 0$.

INDUCTION: $T(n) = \frac{1}{2}T(n-1) + 1$ for integers $n > 1$.

Note that since there are no symbolic constants, this is not a "big-oh" problem; we can solve for $T(n)$ exactly.

a) To begin, what are the initial values of $T(n)$?

$T(1) =$ _____; $T(2) =$ _____; $T(3) =$ _____; $T(4) =$ _____; $T(5) =$ _____.

b) Expand the inductive rule so $T(n)$ is expressed in terms of $T(n-2)$.

---

c) Express $T(n)$ in terms of $T(n-3)$.

---

d) What is the general pattern? That is, express $T(n)$ in terms of $T(n-i)$.

---

e) For what value of i can we eliminate $T(n-i)$ from the expression? $i =$ _____

f) Use your answer to (e) to express $T(n)$ as a function of n alone (i.e., without any terms involving the function T).

$T(n) =$ _____.

Problem 5: (15 points) We say n_0 and c are *witnesses* to the fact that $f(n)$ is $O(g(n))$ if for all $n \geq n_0$ it is true that $f(n) \leq cg(n)$ (i.e., a big-oh proof "works" with n_0 and c as the constants).

a) If $n_0 = 1$, what is the smallest value of c such that n_0 and c are witnesses to the fact that $(n+2)^2$ is $O(n^2)$? $c =$ _____

b) If $c = 5$, what is the smallest nonnegative integer n_0 such that n_0 and c are witnesses to the fact that $(n+2)^2$ is $O(n^2)$? $n_0 =$ _____

c) If $n_0 = 0$, for what values of c are n_0 and c witnesses to the fact that $(n+2)^2$ is $O(n^2)$?

Problem 6: (15 points) Let us consider three-node binary trees whose nodes are each labeled by 3, 5, or 8, and no two nodes have the same label. For the questions below, the "priority" of a label is the same as the value of that label.

a) How many of these trees are balanced, partially-ordered trees? _____ Draw them in the space below.

b) How many such trees have the partially-ordered tree property (POT property), but are *not* balanced? _____ Draw them in the space below.